XX. *The Potential of an Anchor Ring.*—Part II.

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Introduction.

This paper is a continuation of that at pp. 43–95 *suprà*, on "The Potential of an Anchor Ring." In that paper the potential of an anchor ring was found at all external points; in this, its value is determined at internal points. The annular form of rotating gravitating fluid was also discussed in that paper; here the stability of such a ring is considered. In addition, the potential of a ring whose cross-section is elliptic, being of interest in connection with Saturn, is obtained. The similarity of the methods employed, as well as of the analysis, has led me to give in this paper also a determination of the steady motion of a single vortex-ring in an infinite fluid, and of several fine vortex rings on the same axis.

In *Section I.* solutions of Laplace's equation applicable to space inside an anchor ring are obtained. These results are applied to obtain the potential of a solid ring at internal points, and also of a distribution of matter on the surface of the ring. The work done in collecting the ring from infinity is obtained.

In *Section II.* the stability of an annulus of rotating gravitating fluid is considered for three kinds of disturbances.

(1) *Fluted: i.e.*, those in which the ring remains symmetrical about its axis,* but the cross-section is deformed.

(2) *Twisted: i.e.*, those in which the cross-section remains circular, but the circular axis of the ring is deformed.

(3) *Beaded: i.e.*, those in which the circular axis of the ring is undisturbed, but the cross-section is a circle of variable radius.

The ring is found to be stable for fluted and twisted waves, but is broken up by long beaded waves.

* Axis of the ring throughout the paper means the axis of revolution; the central circle of the ring is called the *circular* axis.

Section III. is devoted to Saturn's rings. In LAPLACE's proof ('Méc. Cél.,' Book 3, c. 6) that the rings are not continuous fluid, he assumes the attraction of a ring of elliptic cross-section on a point at the surface to be the same as that of an elliptic cylinder. Mme. KOWALEWSKI, in her memoir on the ring of Saturn ('Astronomische Nachrichten,' No. 2643, vol. 111, 1885) uses a method which applies only to rings of nearly circular section. Here I have attempted to find the potential of a ring of elliptic cross-section. Applied to Saturn, the results obtained agree fairly with LAPLACE's.

In *Section IV.* the steady motion of a single vortex-ring of finite cross-section is discussed. If m be its strength, c its mean radius, and its cross-section be given by

$$\mathrm{R} = a\,\{1 + \beta_2 \cos 2\chi + \beta_3 \cos 3\chi + \beta_4 \cos 4\chi + \ldots\},$$

it is shown that $\beta_2, \beta_3, \beta_4 \ldots$ are of the 2nd, 3rd, &c., orders in a/c, and their values are found as far as $(a/c)^4$.

The velocity of the ring

$$= \frac{m}{2\pi c}\left\{\log\frac{8c}{a} - \frac{1}{4} - \frac{12\log\frac{8c}{a} - 15}{32}\left(\frac{a}{c}\right)^2 \ldots\right\}$$

$$\beta_2 = -\frac{12\mathrm{L} - 17}{32}\left(\frac{a}{c}\right)^2.$$

The results agree with those given by Mr. HICKS, obtained by means of Toroidal Functions. ('Phil. Trans.,' 1884–1885.)

In *Section V.* the motion of a number of fine vortex rings on the same axis is discussed. Equations are obtained giving the forward velocity and the rate of increase of the radius of each ring. Let m_1 be the strength, c_1 the mean radius, a_1 the radius of the cross-section, and z_1 the distance of the centre along the axis of z for one of the rings.

It is shown that the kinetic energy of the system is given by

$$\mathrm{T} = 8\Sigma\left\{\frac{m_1{}^2 c_1}{2}\left(\log\frac{8c_1}{a_1} - \frac{7}{4}\right) + m_1 m_2 \int_0^\pi \frac{c_1 c_2 \cos\phi\, d\phi}{\sqrt{\{(z_2 - z_1)^2 + c_2{}^2 - 2c_1 c_2 \cos\phi + c_1{}^2\}}}\right\}.$$

The equations of motion are

$$\left.\begin{aligned} m_1 c_1 \dot{z}_1 &= \frac{1}{8\pi}\frac{\partial \mathrm{T}}{\partial c_1} \\ -\, m_1 c_1 \dot{c}_1 &= \frac{1}{8\pi}\frac{\partial \mathrm{T}}{\partial z_1} \end{aligned}\right\}.$$

The momentum integral takes the simple form

$$\Sigma\,(m_1 c_1{}^2) = \text{const.}$$

The special cases of one ring pursuing another of equal strength, of the direct approach of a ring towards a fixed plane, and of the motion of a ring over a spherical obstacle are considered in detail.

Section I.

§§ 1–4. Solution of LAPLACE's equation inside an anchor ring.
§§ 5–7. Potential of a solid anchor ring at internal points.
Exhaustion of potential energy.
§§ 8–9. Potential of a distribution of matter on the surface of a ring, at all internal points, and at external points near the ring.

Section II.

§§ 10–13. Potential and exhaustion of potential energy of a ring whose cross-section is $R = a(1 + \Sigma \beta_n \cos n\chi)$.
§§ 14–15. Small fluted oscillations of a gravitating fluid ring.
§§ 16–19. Exhaustion of potential energy of a ring which has beaded waves on it.
§ 20. Such waves render the ring unstable.
§§ 21–22. Exhaustion of potential energy of a ring whose central circle is deformed.
§ 23. Such deformation does not produce instability.

Section III.

§§ 24–26. Potential of a ring, whose cross-section is elliptic, at external points.
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Section IV.

§ 29. Steady motion of a single thick vortex ring.
§§ 30–34. The stream-line function.
§§ 35–36. Determination of the velocity and the form of the cross-section.
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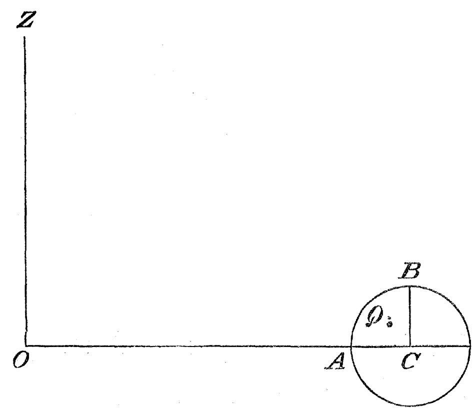
Section V.

§ 38. Stream-line function for any number of fine vortex rings on the same axis.
§ 39. The equations of motion of the rings.
§§ 40–42. The integrals of momentum and energy.
§§ 43–47. Special cases of two rings.

SECTION I.

§ 1. *To find solutions of* LAPLACE'S *Equation applicable to space inside an anchor ring.*



Let O be the centre of the ring, OZ its axis.

Let any section through OZ cut the ring in the circle whose centre is C.

Let Q be any point in that section.

Let $\mathrm{OC} = c$, $\mathrm{CA} = a$, and let the coordinates of Q, referred to CA and CB as axes, be x and z; also let the polar coordinate of Q, referred to C as origin, CA as initial line, be R and χ, so that $\mathrm{CQ} = \mathrm{R}$ and $\angle\, \mathrm{ACQ} = \chi$.

In cylindrical coordinates, LAPLACE'S equation is

$$\frac{d^2\mathrm{V}}{d\varpi^2} + \frac{1}{\varpi}\cdot\frac{d\mathrm{V}}{d\varpi} + \frac{d^2\mathrm{V}}{dz^2} + \frac{1}{\varpi^2}\cdot\frac{d^2\mathrm{V}}{d\phi^2} = 0.$$

Writing $\varpi = c - x$, this becomes

$$\frac{d^2\mathrm{V}}{dx^2} + \frac{d^2\mathrm{V}}{dz^2} = \frac{1}{c - x}\cdot\frac{d\mathrm{V}}{dx} - \frac{1}{(c - x)^2}\cdot\frac{d^2\mathrm{V}}{d\phi^2}.$$

We shall find solutions of this equation in descending powers of c.

First, consider the case where V is independent of ϕ.

Then

$$c\left(\frac{d^2\mathrm{V}}{dx^2} + \frac{d^2\mathrm{V}}{dz^2}\right) = x\left(\frac{d^2\mathrm{V}}{dx^2} + \frac{d^2\mathrm{V}}{dz^2}\right) + \frac{d\mathrm{V}}{dx}.$$

Let

$$\mathrm{V} = \mathrm{U}_n + \frac{1}{c}\,\mathrm{U}_{n+1} + \frac{1}{c^2}\,\mathrm{U}_{n+2} + \&\mathrm{c.},$$

then

$$\left.\begin{aligned}
&\frac{d^2U_n}{dx^2} + \frac{d^2U_n}{dz^2} = 0 \\
&\frac{d^2U_{n+1}}{dx^2} + \frac{d^2U_{n+1}}{dz^2} = x\left(\frac{d^2U_n}{dx^2} + \frac{d^2U_n}{dz^2}\right) + \frac{dU_n}{dx} \\
&\frac{d^2U_{n+2}}{dx^2} + \frac{d^2U_{n+2}}{dz^2} = x\left(\frac{d^2U_{n+1}}{dx^2} + \frac{d^2U_{n+1}}{dz^2}\right) + \frac{dU_{n+1}}{dx}
\end{aligned}\right\}$$

&c., &c.

Let

$$U_n = (x + iz)^n,$$

then

$$\frac{d^2U_{n+1}}{dx^2} + \frac{d^2U_{n+1}}{dz^2} = n\,(x + iz)^{n-1},$$

therefore

$$U_{n+1} = \tfrac{1}{4}(x - iz)\,(x + iz)^n + C\,(x + iz)^{n+1}.$$

Take

$$C = 0,$$

then

$$U_{n+1} = \tfrac{1}{4}(x - iz)\,(x + iz)^n.$$

From this U_{n+2} might be found, and so on.

§ 2. It is, however, more convenient to transform the equations to polar coordinates. Let

$$x = ae^\rho \cos\chi, \qquad z = ae^\rho \sin\chi.$$

For points inside the ring ρ varies from $-\infty$ to 0, and χ from 0 to 2π.

Now,

$$\frac{dU}{dx} = \frac{1}{a}\,e^{-\rho}\left(\cos\chi\,\frac{dU}{d\rho} - \sin\chi\,\frac{dU}{d\chi}\right),$$

$$\frac{dU}{dz} = \frac{1}{a}\,e^{-\rho}\left(\sin\chi\,\frac{dU}{d\rho} + \cos\chi\,\frac{dU}{d\chi}\right),$$

and

$$\frac{d^2U}{dx^2} + \frac{d^2U}{dz^2} = \frac{1}{a^2}\,e^{-2\rho}\left(\frac{d^2U}{d\rho^2} + \frac{d^2U}{d\chi^2}\right).$$

Substituting in the above equations, we find

$$\left.\begin{aligned}
&\frac{d^2U_n}{d\rho^2} + \frac{d^2U_n}{d\chi^2} = 0 \\
&\frac{d^2U_{n+1}}{d\rho^2} + \frac{d^2U_{n+1}}{d\chi^2} = ae^\rho\left\{\cos\chi\left(\frac{d^2U_n}{d\rho^2} + \frac{d^2U_n}{d\chi^2} + \frac{dU_n}{d\rho}\right) - \sin\chi\,\frac{dU_n}{d\chi}\right\} \\
&\frac{d^2U_{n+2}}{d\rho^2} + \frac{d^2U_{n+2}}{d\chi^2} = ae^\rho\left\{\cos\chi\left(\frac{d^2U_{n+1}}{d\rho^2} + \frac{d^2U_{n+1}}{d\chi^2} + \frac{dU_{n+1}}{d\rho}\right) - \sin\chi\,\frac{dU_{n+1}}{d\chi}\right\} \\
&\qquad\qquad \&c, \qquad\qquad \&c,
\end{aligned}\right\} \quad . \quad . \quad (1).$$

Let

$$U_n = e^{n\rho} \cos n\chi.$$

Then

$$\frac{d^2U_{n+1}}{d\rho^2} + \frac{d^2U_{n+1}}{d\chi^2} = ae^{\overline{n+1}\rho} n \cos \overline{n-1}\chi.$$

Therefore

$$U_{n+1} = an \frac{e^{\overline{n+1}\rho} \cos \overline{n-1}\chi}{(n+1)^2 - (n-1)^2},$$

$$= a \tfrac{1}{4} e^{(n+1)\rho} \cos (n-1)\chi.$$

This gives

$$\frac{d^2U_{n+2}}{d\rho^2} + \frac{d^2U_{n+2}}{d\chi^2} = a^2e^{(n+2)\rho} \left\{ \frac{3n}{4} \cos (n-2)\chi + \frac{2n+1}{4} \cos n\chi \right\}.$$

Therefore

$$U_{n+2} = a^2e^{(n+2)\rho} \left\{ \frac{(2n+1)}{16\,(n+1)} \cos n\chi + \tfrac{3}{32} \cos (n-2)\chi \right\}.$$

In the same way we find

$$U_{n+3} = \left(\frac{a}{2}\right)^3 e^{(n+3)\rho} \left\{ \tfrac{1}{2} \frac{(2n+1)\,(2n+3)}{(2n+2)\,(2n+4)} \cos (n+1)\chi + 2\,\frac{1\,.\,3}{2\,.\,4} \frac{2n+1}{2n+2} \cos (n-1)\chi \right.$$
$$\left. + \frac{1\,.\,3\,.\,5}{2\,.\,4\,.\,6} \cos (n-3)\chi \right\}.$$

This suggests the form of U_{n+p}, which is easily verified.

$$U_{n+p} = \left(\frac{a}{2}\right)^p e^{(n+p)\rho} \left\{ \tfrac{1}{2} \frac{(2n+1)\ldots(2n+2p-3)}{(2n+2)\ldots(2n+2p-2)} \cos (n+p-2)\chi \right.$$
$$+ \frac{p-1}{1} \frac{1\,.\,3}{2\,.\,4} \frac{(2n+1)\ldots(2n+2p-5)}{(2n+2)\ldots(2n+2p-4)} \cos (n+p-4)\chi$$
$$\left. + \frac{(p-1)(p-2)}{2\,!} \frac{1.3.5}{2.4.6} \frac{(2n+1)\ldots(2n+2p-7)}{(2n+2)\ldots(2n+2p-6)} \cos (n+p-6)\chi + \ldots \right\} \quad (2).$$

This formula for U_{n+p} holds whether p be $>$ or $<$ n. The number of terms in U_{n+p} is p.

§ 3. The series $U_n + \frac{1}{c} U_{n+1} + \frac{1}{c^2} U_{n+2} + \ldots$ is a solution of LAPLACE's equation. It is convergent at all internal points.

For U_{n+p} is $< \left(\frac{a}{2}\right)^p e^{(n+p)\rho} (1+1)^{p-1}$.

Thus the series $U_n + \frac{1}{c} U_{n+1} \ldots$ converges more rapidly than the geometrical progression

$$1 + \frac{a}{c} + \frac{a^2}{c^2} + \&c.,$$

Writing sines instead of cosines, we obtain another set of solutions of LAPLACE'S equation applicable to space inside a ring.

Writing $-n$ for n solutions of LAPLACE'S equation applicable to space just outside an anchor ring are obtained. They can, however, only be obtained as far as the nth term. At this point log R arises in the series. In obtaining each term from the preceding, as an integration is performed, a constant should be added; and for the series to be convergent these constants must have definite values. To find the series further, the method of my former paper is necessary.

The following are the solutions of LAPLACE'S equation inside a ring for $n = 0$, 1, 2, 3, or 4.

Constant.

$$\left.\begin{aligned}
&\frac{\mathrm{R}}{a}\cos\chi + \frac{a}{2c}\cdot\tfrac{1}{2}\cdot\frac{\mathrm{R}^2}{a^2} + \left(\frac{a}{2c}\right)^2\tfrac{3}{4}\cdot\frac{\mathrm{R}^3}{a^3}\cos\chi + \left(\frac{a}{2c}\right)^3\frac{\mathrm{R}^4}{a^4}\left(\tfrac{5}{8}\cos 2\chi + \tfrac{9}{16}\right)\\
&\qquad + \left(\frac{a}{2c}\right)^4\frac{\mathrm{R}^5}{a^5}\left(\tfrac{35}{128}\cos 3\chi + \tfrac{45}{64}\cos\chi\right) + \ldots,\\
&\frac{\mathrm{R}^2}{a^2}\cos 2\chi + \frac{a}{2c}\cdot\tfrac{1}{2}\cdot\frac{\mathrm{R}^3}{a^3}\cos\chi + \left(\frac{a}{2c}\right)^2\frac{\mathrm{R}^4}{a^4}\left(\tfrac{5}{12}\cos 2\chi + \tfrac{3}{8}\right)\\
&\qquad\quad + \left(\frac{a}{2c}\right)^3\frac{\mathrm{R}^5}{a^5}\left(\tfrac{65}{96}\cos 3\chi + \tfrac{5}{8}\cos\chi\right) + \ldots,\\
&\frac{\mathrm{R}^3}{a^3}\cos 3\chi + \frac{a}{2c}\cdot\tfrac{1}{2}\cdot\frac{\mathrm{R}^4}{a^4}\cos 2\chi + \left(\frac{a}{2c}\right)^2\frac{\mathrm{R}^5}{a^5}\left(\tfrac{65}{96}\cos 3\chi + \tfrac{5}{8}\cos\chi\right) + \ldots,\\
&\frac{\mathrm{R}^4}{a^4}\cos 4\chi + \frac{a}{2c}\cdot\tfrac{1}{2}\cdot\frac{\mathrm{R}^5}{a^5}\cos 3\chi + \ldots
\end{aligned}\right\} \quad . \quad (3).$$

§ 4. In discussing the stability of a fluid ring, approximate solutions of the equation

$$\frac{d^2\mathrm{V}}{dx^2} + \frac{d^2\mathrm{V}}{dx^2} - \frac{1}{c-x}\frac{d\mathrm{V}}{dx} + \frac{1}{(c-x)^2}\frac{d^2\mathrm{V}}{d\phi^2} = 0$$

are required.

Writing V $\cos p\phi$ for V, it becomes

$$\frac{d^2\mathrm{V}}{dx^2} + \frac{d^2\mathrm{V}}{dz^2} - \frac{1}{c-x}\frac{d\mathrm{V}}{dx} - \frac{p^2}{(c-x)^2}\mathrm{V} = 0.$$

If p is of the order c/a the last term becomes of the same order as the terms $\frac{d^2\mathrm{V}}{dx^2} + \frac{d^2\mathrm{V}}{dz^2}$, and the equation may be written

$$\frac{d^2\mathrm{V}}{dx^2} + \frac{d^2\mathrm{V}}{dz^2} - \frac{p^2}{c^2}\mathrm{V} - \frac{1}{c}\left\{\frac{d\mathrm{V}}{dx} + \frac{2p^2}{c^2}x\mathrm{V}\right\} - \frac{1}{c^2}\left\{x\frac{d\mathrm{V}}{dx} + \frac{3p^2}{c^2}x^2\mathrm{V}^2\right\} - \&\mathrm{c}. = 0$$

and the equation must be regarded as an approximation to

$$\frac{d^2V}{dx^2} + \frac{d^2V}{dz^2} - \frac{p^2}{c^2} V = 0$$

or

$$\frac{d^2V}{dR^2} + \frac{1}{R} \cdot \frac{dV}{dR} + \frac{1}{R^2} \frac{d^2V}{d\chi^2} - \frac{p^2}{c^2} V = 0,$$

of which $\cos n\chi \,.\, J_n\left(\frac{ipR}{c}\right)$ is a solution; and starting from this, an approximate solution may be obtained.

We shall only need the cases in which p is small.

The equation may then be written

$$\frac{d^2V}{dx^2} + \frac{d^2V}{dz^2} = \frac{1}{c}\frac{dV}{dx} + \frac{1}{c^2}\left(x\frac{dV}{dx} + p^2 V\right) + \frac{1}{c^3}\left(x^2\frac{dV}{dx} + 2p^2 xV\right) + \ldots$$

An approximate solution of this is

$$\cos p\phi \left\{\frac{R^n}{a^n}\cos n\chi + \frac{a}{4c}\frac{R^{n+1}}{a^{n+1}}\cos(n-1)\chi \right.$$
$$\left. + \frac{a^2}{8c^2}\frac{R^{n+2}}{a^{n+2}}\left(\frac{4p^2+2n+1}{2n+2}\cos n\chi + 3\cos(n-2)\chi\right) + \&c.\right\} \quad . \quad . \quad . \quad (4).$$

When $n = 0$ the solution is

$$\cos p\phi\left\{1 + \frac{a^2}{4c^2}p^2\frac{R^2}{a^2} + \&c.\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5).$$

§ 5. *To find the Potential of a solid anchor ring at an internal point.*

At any external point whose polar coordinates are r, θ, ϕ,

$$V = \frac{M}{\pi}\left\{\int_0^\pi \frac{d\phi}{\sqrt{(r^2+c^2-2cr\sin\theta\cos\phi)}} + \frac{a^2}{8}\cdot\frac{d}{cdc}\int_0^\pi \frac{d\phi}{\sqrt{(r^2+c^2-2cr\sin\theta\cos\phi)}} \right.$$
$$\left. - \frac{a^4}{192}\left(\frac{d}{cdc}\right)^2\int_0^\pi \frac{d\phi}{\sqrt{(r^2+c^2-2cr\sin\theta\cos\phi)}} - \&c.\right\}.*$$

Using the expansions of the integrals given in that paper, we find that at the surface of the ring

* *Suprà*, p. 59.

$$V = 2\pi a^2 \left\{\lambda + 2 + \frac{\lambda + \frac{3}{4}}{2}\sigma\cos\chi + \left[\frac{\lambda - \frac{1}{2}}{16} + \frac{3(\lambda + \frac{17}{36})}{16}\cos 2\chi\right]\sigma^2 \right.$$
$$+ \left[\frac{3(\lambda - \frac{1}{12})}{32}\cos\chi + \frac{5(\lambda + \frac{7}{24})}{64}\cos 3\chi\right]\sigma^3$$
$$\left. + \left[\frac{9\lambda}{256} + \frac{7(\lambda - \frac{19}{168})}{128}\cos 2\chi + \frac{35(\lambda + \frac{19}{120})}{1024}\cos 4\chi\right]\sigma^4 + \&c.\right\}. \quad (6),$$

where $\sigma = a/c$, and $\lambda = \log(8c/a) - 2$.

Now, inside the ring,

$$\frac{d^2V}{dz^2} + \frac{d^2V}{d\varpi^2} + \frac{1}{\varpi}\frac{dV}{d\varpi} + 4\pi = 0.$$

Let

$$V = -2\pi z^2 + V_1,$$

then

$$\frac{d^2V_1}{dz^2} + \frac{d^2V_1}{d\varpi^2} + \frac{1}{\varpi}\frac{dV_1}{d\varpi} = 0.$$

We may, therefore, assume that

$$\frac{V}{2\pi a^2} = -\frac{z^2}{a^2} + A_1$$
$$+ A_2\left\{\frac{R}{a}\cos\chi + \frac{\sigma}{4}\frac{R^2}{a^2} + \frac{3\sigma^2}{16}\frac{R^3}{a^3}\cos\chi + \sigma^3\left(\frac{R}{a}\right)^4\left(\tfrac{5}{64}\cos 2\chi + \tfrac{9}{128}\right) + \ldots\right\},$$
$$+ A_3\left\{\frac{R^2}{a^2}\cos 2\chi + \frac{\sigma}{4}\frac{R^3}{a^3}\cos\chi + \sigma^2\frac{R^4}{a^4}\left(\tfrac{5}{48}\cos 2\chi + \tfrac{3}{32}\right)\right.$$
$$+ \sigma^3\frac{R^5}{a^5}\left(\tfrac{35}{768}\cos 3\chi + \tfrac{15}{128}\cos\chi\right)$$
$$\left. + \sigma^4\frac{R^6}{a^6}\left(\tfrac{21}{1024}\cos 4\chi + \tfrac{35}{512}\cos 2\chi + \tfrac{25}{512}\right) + \ldots\right\},$$
$$+ A_4\left\{\frac{R^3}{a^3}\cos 3\chi + \frac{\sigma}{4}\frac{R^4}{a^4}\cos 2\chi + \ldots\right\},$$
$$+ A_5\left\{\frac{R^4}{a^4}\cos 2\chi + \ldots\right\}.$$

Therefore, at the surface of the ring,

$$\frac{V}{2\pi a^2} = -\tfrac{1}{2} + A_1 + A_2\left(\frac{\sigma}{4} + \frac{9\sigma^3}{128}\right) + A_3\left(\frac{3\sigma^2}{32} + \frac{25\sigma^4}{512}\right),$$
$$+ \cos\chi\left\{A_2\left(1 + \frac{3\sigma^2}{16}\right) + A_3\left(\frac{\sigma}{4} + \frac{15\sigma^3}{128}\right)\right\},$$
$$+ \cos 2\chi\left\{\tfrac{1}{2} + A_2\frac{5\sigma^3}{64} + A_3\left(1 + \frac{5\sigma^2}{48} + \frac{35\sigma^4}{512}\right) + A_4\frac{\sigma}{4}\right\},$$
$$+ \cos 3\chi\left\{A_3\frac{35\sigma^3}{768} + A_4\right\},$$
$$+ \cos 4\chi\left\{A_3\frac{21\sigma^4}{1024} + A_5\right\} \quad \ldots \ldots \quad (7).$$

Comparing this with the value of the potential at the surface of the ring, given above, we have equations to determine the constants A_1, A_2, &c.

These give

$$\left.\begin{aligned} A_1 &= +\lambda\tfrac{5}{2} - \frac{\lambda+\frac{7}{4}}{16}\sigma^2 - \frac{3(\lambda+\frac{3}{4})}{512}\sigma^4 \\ A_2 &= \frac{\lambda+1}{2}\sigma - \frac{3\lambda+5}{64}\sigma^3 \\ A_3 &= -\tfrac{1}{2} + \frac{3(\lambda+\frac{3}{4})}{16}\sigma^2 - \frac{12\lambda+19}{512}\sigma^4 \\ A_4 &= \frac{5(\lambda+\frac{7}{12})}{64}\sigma^3 \\ A_5 &= \frac{35(\lambda+\frac{11}{24})}{1024}\sigma^4 \end{aligned}\right\} \quad \ldots\ldots\ldots (8).$$

Thus the potential is found at all internal points as far as terms in σ^4.

Writing L ($\log 8c/a$) instead of $\lambda + 2$, as far as terms of the second order

$$V = 2\pi a^2\left\{L + \tfrac{1}{2}\left(1 - \frac{R^2}{a^2}\right) + \sigma\left[\frac{L-1}{2}\frac{R}{a} - \frac{R^3}{8a^3}\right]\cos\chi\right.$$

$$\left. + \sigma^2\left[-\frac{L-\frac{1}{4}}{16} + \frac{L-1}{8}\frac{R^2}{a^2} - \tfrac{3}{64}\frac{R^4}{a^4} + \frac{3(L-\frac{5}{4})}{16}\frac{R^2}{a^2}\cos 2\chi - \tfrac{5}{96}\frac{R^4}{a^4}\cos 2\chi\right] + \ldots\right\} \quad (9).$$

§ 6. The value of V is a maximum, when

$$\chi = 0,$$

and

$$-2R + \sigma\left[(L-1)a - \tfrac{3}{4}\frac{R^2}{a}\right] = 0,$$

that is, when $R = \sigma\,.\,\tfrac{1}{2}(L-1)a +$ terms in σ^3.

This maximum value

$$= 2\pi a^2\left\{L + \tfrac{1}{2} - \frac{\sigma^2}{8}(L-1)^2 + \frac{\sigma^2}{4}(L-1)^2 - \ldots\right\}$$

$$= 2\pi a^2\left\{L + \tfrac{1}{2} + \frac{\sigma^2}{8}(L-1)^2 + \ldots\right\}.$$

§ 7. The exhaustion of potential energy

$$= \tfrac{1}{2}\iiint \mathrm{V}\, dm$$

$$= \tfrac{1}{2}\,.\,2\pi\int_0^a\int_0^{2\pi}\mathrm{V}\,(c - \mathrm{R}\cos\chi)\,\mathrm{R}\,d\mathrm{R}\,d\chi$$

$$= 4\pi^3 a^2 c\int_0^a\left\{\mathrm{A}_1 - \frac{\mathrm{R}^2}{2a^2} + \mathrm{A}_2\left(\frac{\sigma}{4}\frac{\mathrm{R}^2}{a^2} + \frac{9\sigma^3}{128}\frac{\mathrm{R}^4}{a^4}\right) + \mathrm{A}_3\left(\frac{3\sigma^2}{32}\frac{\mathrm{R}^4}{a^4} + \frac{25\sigma^4}{512}\frac{\mathrm{R}^6}{a^6}\right)\right\}\mathrm{R}\,d\mathrm{R}$$

$$- 2\pi^3 a^2\int_0^a\left\{\mathrm{A}_2\left(\frac{\mathrm{R}}{a} + \frac{3\sigma^2}{16}\frac{\mathrm{R}^3}{a^3}\right) + \mathrm{A}_3\left(\frac{\sigma}{4}\frac{\mathrm{R}^3}{a^3} + \tfrac{15}{128}\sigma^3\frac{\mathrm{R}^5}{a^5}\right)\right\}\mathrm{R}^2\,d\mathrm{R}$$

$$= 2\pi^3 a^4 c\left\{-\tfrac{1}{4} + \mathrm{A}_1 - \mathrm{A}_2\left(\frac{\sigma}{8} + \frac{\sigma^3}{128}\right) - \mathrm{A}_3\left(\frac{\sigma^2}{96} - \frac{5\sigma^4}{1024}\right)\right\}$$

$$= \frac{\mathrm{M}^2}{2\pi c}\left\{\mathrm{L} + \tfrac{1}{4} - \frac{\mathrm{L} - \frac{2}{3}}{8}\sigma^2 - \frac{3\,(\mathrm{L} - \frac{19}{12})}{512}\sigma^4 - \ldots\right\} \quad \ldots \ldots \ldots \quad (10).$$

If we may judge from these few terms this series is very convergent. When $\sigma = 1$, *i.e.*, in the case of a ring so thick that it touches itself at the origin, we have $\mathrm{L} = \log_e 8 = 2{\cdot}080$.

The exhaustion of energy

$$= \frac{\mathrm{M}^2}{2\pi c}\,.\,\{2{\cdot}330 - {\cdot}177 - {\cdot}003\}$$

$$= 2{\cdot}150\,.\,\frac{\mathrm{M}^2}{2\pi c}.$$

The first three figures would seem to be correct.

[Let the same mass be collected into rings of different mean radii, and consequently different thicknesses; let the final mean radius of the ring, that is, the mean radius when the ring is so thick that it touches itself at its centre, be taken as unity; let the exhaustion of energy in this position be also taken as unity. Then the exhaustion of energy for different mean radii of the ring is given by the following Table :—

Mean radius	1·25	1·5	1·75	2	3	4	5
Exhaustion of potential energy	·9488	·8874	·8284	·7749	·6144	·5162	·3506

July 22, 1892.]*

* This Table was given at the suggestion of one of the Referees, replacing a Table given in the paper as read.

§ 8. *To find the potential of a distribution of matter of density $\beta_n \cos n\chi$ on the surface of an anchor ring, at all internal points and at external points near the surface of the ring.*

We know that for a cylinder

$$\left.\begin{aligned} \frac{nV_i}{2\pi\beta_n} &= \left(\frac{R}{a}\right)^n \cos n\chi \\ \text{and} \qquad \frac{nV_0}{2\pi\beta_n} &= \left(\frac{a}{R}\right)^n \cos n\chi \end{aligned}\right\}.$$

Let us assume for a ring

$$\begin{aligned} \frac{nV_i}{2\pi\beta_n} = \left(\frac{R}{a}\right)^n \cos n\chi \\ &+ \frac{\sigma}{4}\left(\frac{R}{a}\right)^{n+1} \cos(n-1)\chi + \frac{\sigma^2}{8}\left(\frac{R}{a}\right)^{n+2}\left\{\frac{2n+1}{2n+2}\cos n\chi + \tfrac{3}{4}\cos(n-2)\chi\right\} + \dots \\ &+ \frac{\sigma}{4}A_1\left\{\left(\frac{R}{a}\right)^{n-1}\cos(n-1)\chi + \frac{\sigma}{4}\left(\frac{R}{a}\right)^n \cos(n-2)\chi + \dots\right\} \\ &+ \frac{\sigma}{4}A_2\left\{\left(\frac{R}{a}\right)^{n+1}\cos(n+1)\chi + \frac{\sigma}{4}\left(\frac{R}{a}\right)^n \cos n\chi + \dots\right\} \\ &+ \frac{\sigma^2}{16}A_3\left\{\left(\frac{R}{a}\right)^{n-2}\cos(n-2)\chi + \dots\right\} \\ &+ \frac{\sigma^2}{16}A_4\left\{\left(\frac{R}{a}\right)^n \cos n\chi + \dots\right\} \\ &+ \frac{\sigma^2}{16}A_5\left\{\left(\frac{R}{a}\right)^{n-2}\cos(n+2)\chi + \dots\right\} \end{aligned}$$

and

$$\begin{aligned} \frac{nV_0}{2\pi\beta_n} = \left(\frac{a}{R}\right)^n \cos n\chi \\ &+ \frac{\sigma}{4}\left(\frac{a}{R}\right)^{n-1}\cos(n+1)\chi + \frac{\sigma^2}{8}\left(\frac{a}{R}\right)^{n-2}\left\{\frac{2n-1}{2n-2}\cos(n-2)\chi + \tfrac{3}{4}\cos(n-2)\chi\right\} + \dots \\ &+ \frac{\sigma}{4}B_1\left\{\left(\frac{a}{R}\right)^{n-1}\cos(n-1)\chi + \frac{\sigma}{4}\left(\frac{a}{R}\right)^{n-2}\cos n\chi + \dots\right\} \\ &+ \frac{\sigma}{4}B_2\left\{\left(\frac{a}{R}\right)^{n+1}\cos(n+1)\chi + \frac{\sigma}{4}\left(\frac{a}{R}\right)^n \cos(n+2)\chi + \dots\right\} \\ &+ \frac{\sigma^2}{16}B_3\left\{\left(\frac{a}{R}\right)^{n-2}\cos(n-2)\chi + \dots\right\} \\ &+ \frac{\sigma^2}{16}B_4\left\{\left(\frac{a}{R}\right)^n \cos n\chi + \dots\right\} \\ &+ \frac{\sigma^2}{16}B_5\left\{\left(\frac{a}{R}\right)^{n+2}\cos(n+2)\chi + \dots\right\}. \end{aligned}$$

The constants are to be chosen so that

$$\left.\begin{aligned} &V_i - V_0 = 0 \\ \text{and}\qquad &\frac{dV_0}{dR} - \frac{dV_i}{dR} + 4\pi\beta_n \cos n\chi = 0 \end{aligned}\right\}$$

at the surface of the ring.

Thus

$$\left.\begin{aligned} 1 + A_1 &= B_1 \\ (n+1) + (n-1) A_1 &= -(n-1) B_1 \\ A_2 &= 1 + B_2 \\ (n+1) A_2 &= -(n-1) - (n+1) B_2 \end{aligned}\right\}$$

and

$$\left.\begin{aligned} \tfrac{3}{2} + A_1 + A_3 &= B_3 \\ \tfrac{3}{2}(n+2) + nA_1 + (n-2) A_3 &= -(n-2) B_3 \\ \frac{2n+1}{n+1} + A_2 + A_4 &= \frac{2n-1}{n-1} + B_1 + B_4 \\ \frac{2n+1}{n+1}(n+2) + (n+2) A_2 + nA_4 &= -\frac{2n-1}{n-1}(n-2) - (n-2) B_1 - nB_4 \\ A_5 &= \tfrac{3}{2} + B_2 + B_5 \\ (n+2) A_5 &= -\tfrac{3}{2}(n-2) - nB_2 - (n+2) B_5 \end{aligned}\right\}$$

These equations give

$$A_1 = -\frac{n}{n-1}. \qquad B_1 = -\frac{1}{n-1}.$$

$$A_2 = \frac{1}{n+1}. \qquad B_2 = -\frac{n}{n+1}.$$

$$A_3 = -\frac{n}{2(n-2)}. \qquad B_3 = -\frac{(2n-3)}{(n-1)(n-2)}.$$

$$A_4 = -2. \qquad B_4 = -2.$$

$$A_5 = \frac{2n+3}{(n+1)(n+2)}. \qquad B_5 = -\frac{n}{2(n+2)}.$$

Thus we find that to the second power of σ,

$$\begin{aligned} \frac{Vi}{2\pi a\beta_n} = \left(\frac{R}{a}\right)^n \frac{\cos n\chi}{n} & \\ + \frac{\sigma}{4}\Bigg[\left\{\frac{1}{n}\left(\frac{R}{a}\right)^{n+1} - \frac{1}{n-1}\left(\frac{R}{a}\right)^{n-1}\right\} & \cos(n-1)\chi + \frac{1}{n(n+1)}\left(\frac{R}{a}\right)^{n+1} \cos(n+1)\chi\Bigg] \\ + \frac{\sigma^2}{32}\Bigg[\frac{2(2n+3)}{n(n+1)(n+2)}\left(\frac{R}{a}\right)^{n+2} & \cos(n+2)\chi + \frac{4}{n}\left\{\left(\frac{R}{a}\right)^{n+2} - \left(\frac{R}{a}\right)^n\right\} \cos n\chi \\ + \left\{\frac{3}{n}\left(\frac{R}{a}\right)^{n+2} - \frac{2}{n-1}\left(\frac{R}{a}\right)^n - \frac{1}{n-2}\left(\frac{R}{a}\right)^{n-2}\right\} & \cos(n-2)\chi\Bigg]. \quad (11), \end{aligned}$$

and

$$\frac{V_0}{2\pi a\beta_n} = \left(\frac{a}{R}\right)^n \frac{\cos n\chi}{n}$$
$$+ \frac{\sigma}{4}\left[\left\{\frac{1}{n}\left(\frac{a}{R}\right)^{n-1} - \frac{1}{n+1}\left(\frac{a}{R}\right)^{n+1}\right\}\cos(n+1)\chi - \frac{1}{n(n-1)}\left(\frac{a}{R}\right)^{n-1}\cos(n-1)\chi\right]$$
$$+ \frac{\sigma^2}{32}\left[-\frac{2(2n-3)}{n(n-1)(n-2)}\left(\frac{a}{R}\right)^{n-2}\cos(n-2)\chi + \frac{4}{n}\left\{\left(\frac{a}{R}\right)^{n-2} - \left(\frac{a}{R}\right)^n\right\}\cos n\chi\right.$$
$$\left. + \left\{\frac{3}{n}\left(\frac{a}{R}\right)^{n-2} - \frac{2}{n+1}\left(\frac{a}{R}\right)^n - \frac{1}{n+2}\left(\frac{a}{R}\right)^{n+2}\right\}\cos(n+2)\chi\right] \quad (11a).$$

These agree in giving at the surface of the ring

$$\frac{V}{2\pi a\beta_n} = \frac{\cos n\chi}{n} + \frac{\sigma}{4n}\left\{\frac{\cos(n+1)\chi}{n+1} - \frac{\cos(n-1)\chi}{n-1}\right\}$$
$$+ \frac{\sigma^2}{16n}\left\{\frac{2n+3}{(n+1)(n+2)}\cos(n+2)\chi - \frac{2n-3}{(n-1)(n-2)}\cos(n-2)\chi\right\} + \&c. \quad (12).$$

I have verified the above value of the potential at external points near the ring by evaluating the integral. The method here given is suggested by LAPLACE ('Méc. Cél.,' Book 3, c. 6). It is not satisfactory, as it will not give the value of V further than the terms in σ^n. The form of the potential at points near the ring changes after these terms. Log R is introduced, and it is found that some of the equations coincide so that there are not enough to determine the necessary constants.

§ 9. [As illustrating this, and furnishing a result which will be used later, consider a distribution of matter of density $a\beta_2 \cos 2\chi$ on the surface of the ring.

By § 22 of my previous paper, it is seen that

$$\frac{V_0}{2\pi a^2\beta_2} = \tfrac{1}{2}\frac{a^2}{R^2}\cos 2\chi + \frac{\sigma}{4}\left\{\left(\tfrac{1}{2}\frac{a}{R} - \tfrac{1}{3}\frac{a^3}{R^3}\right)\cos 3\chi - \tfrac{1}{2}\frac{a}{R}\cos\chi\right\}$$
$$+ \frac{\sigma^2}{32}\left\{-\frac{4\log\frac{8c}{R}+1}{2} + 2\left(1-\frac{a^2}{R^2}\right)\cos 2\chi + \left(\tfrac{3}{2} - \tfrac{2}{3}\frac{a^2}{R^2} - \tfrac{1}{4}\frac{a^4}{R^4}\right)\cos 4\chi\right\} + \&c.$$

This agrees with (11a), except for the term

$$-\frac{\sigma^2}{32}\,\frac{4\log\frac{8c}{R}+1}{2}.$$

The value of the potential, inside, is consequently given by

$$\frac{V_i}{2\pi a^2\beta_2} = \tfrac{1}{2}\frac{R^2}{a^2}\cos 2\chi + \frac{\sigma}{4}\left\{\left(\tfrac{1}{2}\frac{R^3}{a^3} - \frac{R}{a}\right)\cos\chi + \tfrac{1}{6}\frac{R^3}{a^3}\cos 3\chi\right\}$$
$$+ \frac{\sigma^2}{32}\left\{\tfrac{7}{12}\frac{R^4}{a^4}\cos 4\chi + 2\left(\frac{R^4}{a^4} - \frac{R^2}{a^2}\right)\cos 2\chi + \tfrac{3}{2}\frac{R^4}{a^4} - 2\frac{R^2}{a^2} - 2\log\frac{8c}{a}\right\} + \&c.$$

Adding to this the value given in (9) for the potential inside the ring $R = a$, we have correct to the first power of β_2, the potential inside $R = a(1 + \beta_2 \cos 2\chi)$.

The terms in β_2 of the exhaustion of potential energy of this ring are given by

$$2\pi^2 a^2 \int_0^{2\pi}\int_0^{a(1+\beta_2\cos 2\chi)} \left\{ \sigma\left(\frac{L-1}{2}\frac{R}{a} - \frac{R^3}{8a^3}\right)\cos\chi \right.$$

$$\left. + \sigma^2\left[\tfrac{3}{16}(L - \tfrac{5}{4})\frac{R^2}{a^2} - \tfrac{5}{96}\frac{R^4}{a^4}\right]\cos 2\chi\right\}(c - R\cos\chi)\, R\, dR\, d\chi$$

$$+ 2\pi^2 a^2 \beta_2 \int_0^{2\pi}\int_0^{a}\left\{\frac{\sigma}{4}\left(\frac{R^3}{2a^3} - \frac{R}{a}\right)\cos\chi + \frac{\sigma^2}{32}\left(\frac{3}{2}\frac{R^4}{a^4} - 2\frac{R^2}{a^2} - 2L\right)\right\}(c - R\cos\chi)\, R\, dR\, d\chi,$$

all the other terms vanishing on integration.

Therefore this part of the exhaustion of energy

$$= 2\pi^2 a^4 c^2\ \sigma^2\beta_2 \int_0^{2\pi}\left\{\left[-\frac{4L-5}{16} + \tfrac{3}{16}(L - \tfrac{5}{4}) - \tfrac{5}{96}\right]\cos^2 2\chi + \tfrac{1}{24}\cos^2\chi - \tfrac{1}{32}(L + \tfrac{1}{4})\right\} d\chi$$

$$= -\ 2\pi^3 a^4 c\ \beta_2 \frac{\sigma^2}{8}(L - \tfrac{5}{12}) \quad \ldots\ldots \quad (13).$$

July 22, 1893.]

Section II.

§ 10. An annular form is possible for gravitating fluid rotating round an axis in relative equilibrium. When the cross-section of the annulus is small compared with its radius, the cross-section is nearly circular.

Let the ring be disturbed so that the cross-section takes the form given by the equation

$$\rho = a\{1 + \beta_2\cos 2\chi + \ldots + \beta_n \cos n\chi + \ldots\}.$$

We shall prove that the ring is stable for disturbances of this kind, and find the periods of the various oscillations.

It is necessary to find the exhaustion of potential energy. Let this be called U.

Then

$$U = \tfrac{1}{2}\int V_i\, dm$$

$$= \pi \int_0^{\pi}\int_0^{\rho} V_i (c - R\cos\chi)\, R\, dR\, d\chi$$

where V_i is the potential at an internal point.

It is necessary to know U as far as the second order of the small quantities β_2, &c., β_n. Suppose V_i is obtained in the form $v_0 + \Sigma v_n \cos n\chi$, where v_0, v_n are functions of R. Then it is necessary to know v_0 and v_1 to the second order, but $v_2 \ldots v_n \ldots$ only to the first order.

Now

$$\frac{V_i}{2\pi a^2} = \log\frac{8c}{a} + \tfrac{1}{2}\left(1 - \frac{R^2}{a^2}\right) + \sigma\left(\frac{\log 8c/a - 1}{2}\frac{R}{a} - \frac{R^3}{8a^3}\right)\cos\chi + \&c.$$

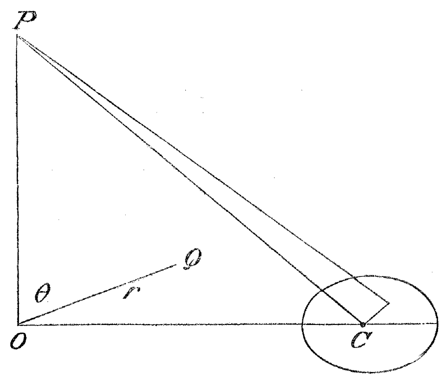
$$+ \Sigma\,\beta_n\left\{\left(\frac{R}{a}\right)^n\frac{\cos n\chi}{n} + \frac{\sigma}{4}\left[\left(\frac{1}{n}\frac{R^{n+1}}{a^{n+1}} - \frac{1}{n-1}\frac{R^{n-1}}{a^{n-1}}\right)\cos(n-1)\chi\right.\right.$$

$$\left.\left. + \frac{1}{n(n+1)}\frac{R^{n+1}}{a^{n+1}}\cos(n+1)\chi\right]\right\}$$

$$+ A + B\left(\frac{R}{a}\cos\chi + \frac{\sigma}{4}\frac{R^2}{a^2} + \&c.\right),$$

where A and B are small quantities of the second order in $\beta_2 \ldots \beta_n \ldots$, which must be found by comparing the value V_i at the surface of the ring with the value of V_0 there.

§ 11. We shall now find V_0 the potential of the ring at an external point.



Let Q be an external point whose coordinates are r and θ.

Let P be a point on the axis, C be the centre of gravity of the cross-section. Let $PCO = \alpha$ and let $CP = r_1$.

The potential at P is

$$2\pi\int_0^{2\pi}\int_0^{\rho}\frac{(c - R\cos\chi)\,R\,dR\,d\chi}{\sqrt{\{r_1^2 + R^2 - 2Rr_1\cos(\chi-\alpha)\}}}$$

$$= 2\pi\int_0^{2\pi}\left\{\frac{c}{r_1}\frac{\rho^2}{2} + \frac{c}{r_1^2}\frac{\rho^3}{3}\cos(\chi-\alpha) + \frac{c}{r_1^3}\frac{\rho^4}{4}P_2[\cos(\chi-\alpha)] - \frac{1}{r_1}\frac{\rho^3}{3}\cos\chi\right.$$

$$\left. - \frac{c\rho^4}{4\rho_1^2}\cos\chi\cos(\chi-\alpha)\ldots\right\}d\chi.$$

Substituting for $\cos\alpha$, c/r_1, and for ρ^p,

$$a^p\left\{1 + p\,\Sigma\,\beta_n\cos n\chi + \frac{p(p-1)}{2}\,\Sigma\left(\frac{\beta_n^{\,2}}{2} + \beta_n\beta_{n+1}\cos\chi\right)\right\},$$

and retaining only for the present the terms of the second degree in $\beta_2 \ldots \beta_n \ldots$, we find that at P

$$V_0 = 4\pi^2a^2c\left\{\frac{1}{2r_1}\,\Sigma\left(\frac{\beta_n^{\,2}}{2} - \sigma\,\beta_n\beta_{n+1}\right) + \frac{ac}{r_1^{\,3}}\,\Sigma\,\frac{\beta_n\beta_{n+2}}{2}\right\}.$$

Therefore at Q

$$V_0 = 2\pi a^2\,\Sigma\left(\frac{\beta_n^{\,2}}{2} - \sigma\,\beta_n\beta_{n+1}\right)\int_0^\pi\frac{c\,d\phi}{\sqrt{(r^2+c^2-2cr\sin\theta\cos\phi)}}$$

$$- 2\pi a^3c\,\Sigma\,(\beta_n\beta_{n+1})\,\frac{d}{dc}\int_0^\pi\frac{d\phi}{\sqrt{(r^2+c^2-2cr\sin\theta\cos\phi)}}.$$

[*Supra*, p. 59.]

Expanding these integrals, we find that at a point R, χ just outside the ring,

$$\frac{V_0}{2\pi a^2} = \Sigma\left(\frac{\beta_n^{\,2}}{2} - \sigma\,\beta_n\beta_{n+1}\right)\left(\log\frac{8c}{R} + \frac{\log\frac{8c}{R} - 1}{2}\,\frac{R}{c}\cos\chi\right)$$

$$+ \sigma\,\Sigma\,\beta_n\beta_{n+1}\left(\frac{c}{R}\cos\chi + \frac{2\log\frac{8c}{R} - 1}{4} + \ldots\right).$$

Adding this to the values already obtained for the potential of a ring whose cross-section is an exact circle of radius a, and to that for a distribution of surface density $a\,\Sigma\,\beta_n\cos n\chi$ on such a ring, we find that

$$\frac{V_0}{2\pi a^2} = \log\frac{8c}{R} + \frac{\log\frac{8c}{R} - 1}{2}\,\frac{R\cos\chi}{c} + \ldots - \frac{\sigma^2}{8}\left(\frac{c}{R}\cos\chi + \frac{2\log\frac{8c}{R} - 1}{4} + \ldots\right)$$

$$+ \Sigma\,\beta_n\left\{\left(\frac{a}{R}\right)^n\frac{\cos n\chi}{n} + \frac{\sigma}{4}\left[\left(\frac{a}{R}\right)^{n-1}\frac{\cos(n+1)\chi}{n}\right.\right.$$

$$\left.\left. - \left(\frac{a}{R}\right)^{n+1}\frac{\cos(n+1)\chi}{n+1} - \left(\frac{a}{R}\right)^{n-1}\frac{\cos(n-1)\chi}{n(n-1)}\right] + \ldots\right\}$$

$$+ \Sigma\left(\frac{\beta_n^{\,2}}{2} - \sigma\,\beta_n\beta_{n+1}\right)\left(\log\frac{8c}{R} + \frac{\log\frac{8c}{R} - 1}{2}\,\frac{R}{c}\cos\chi\right)$$

$$+ \sigma\,\Sigma\,\beta_n\beta_{n+1}\left(\frac{c}{R}\cos\chi + \frac{2\log\frac{8c}{R} - 1}{4} + \ldots\right) \quad \ldots \ldots \ldots \quad (14).$$

§ 12. Now when $R = \rho = a(1 + \Sigma\beta_n \cos n\chi)$, V_i and V_0 must have the same value. This will enable us to find the terms of the second order in V_i.

Remembering that for a solid ring of circular section, V and dV/dR are continuous at the surface, and $d^2V_0/dR^2 = (d^2V_i/dR^2) + 4\pi$, and that for a distribution of surface density $a\beta_n \cos n\chi$

$$\frac{dV_0}{dR} - \frac{dV_i}{dR} + 4\pi a\beta_n \cos n\chi = 0\,,$$

we see that when $R = \rho = a(1 + \Sigma\beta_n \cos n\chi)$

$$\begin{aligned}\frac{V_0 - V_i}{2\pi a^2} &= \frac{4\pi}{2\pi a^2}\frac{a^2}{2}(\beta_2 \cos 2\chi + \ldots + \beta_n \cos n\chi + \ldots)^2 \\ &\quad - \frac{4\pi a\,\Sigma\beta_n \cos n\chi}{2\pi a^2}\,a(\beta_2 \cos 2\chi + \ldots + \beta_n \cos n\chi + \ldots) \\ &\quad + \Sigma\left(\frac{\beta_n^2}{2} - \sigma\beta_n\beta_{n+1}\right)\left(\log\frac{8c}{a} + \frac{\log\frac{8c}{a} - 1}{2}\sigma\cos\chi\right) \\ &\quad + \Sigma\beta_n\beta_{n+1}\left(\cos\chi + \frac{2\log\frac{8c}{a} - 1}{4}\sigma\right) \\ &\quad - A - B\left(\cos\chi + \frac{\sigma}{4}\right)\end{aligned}$$

Since $V_0 - V_i = 0$, we find

$$\left.\begin{aligned}A + B\frac{\sigma}{4} &= (L-1)\,\Sigma\frac{\beta_n^2}{2} - \frac{2L+1}{4}\sigma\,\Sigma\beta_n\beta_{n+1} \\ B &= -\frac{L-1}{4}\sigma\,\Sigma\beta_n^2\end{aligned}\right\}$$

Therefore

$$\begin{aligned}\frac{V_i}{2\pi a^2} &= L + \tfrac{1}{2}\left(1 - \frac{R^2}{a^2}\right) + \sigma\left(\frac{L-1}{2}\frac{R}{a} - \frac{R^3}{8a^3}\right)\cos\chi + \&c. \\ &\quad + \Sigma\beta_n\left\{\frac{1}{n}\frac{R^n}{a^n}\cos n\chi + \frac{\sigma}{4}\left[\left(\frac{1}{n}\frac{R^{n+1}}{a^{n+1}} - \frac{1}{n-1}\frac{R^{n-1}}{a^{n-1}}\right)\cos(n-1)\chi \right.\right. \\ &\qquad\qquad \left.\left. + \frac{1}{n(n+1)}\frac{R^{n+1}}{a^{n+1}}\cos(n+1)\chi\right]\right\} \\ &\quad + (L-1)\,\Sigma\frac{\beta_n^2}{2} - \frac{2L+1}{4}\sigma\,\Sigma\beta_n\beta_{n+1} - \frac{L-1}{16}\sigma^2\,\Sigma\beta_n^2 \\ &\qquad + \frac{L-1}{4}\sigma\,\Sigma\beta_n^2\left(\frac{R}{a}\cos\chi + \frac{\sigma}{4}\frac{R^2}{a^2}\right)\quad \ldots\ldots\ldots\ldots \quad (15.)\end{aligned}$$

§ 13. The exhaustion of potential energy is given by

$$U = \tfrac{1}{2}\int V_i \, dm$$

$$= \pi a^2 \iiint \left\{ L + \tfrac{1}{2}\left(1 - \frac{R^2}{a^2}\right) + \sigma \cos\chi \left(\frac{L-1}{2}\frac{R}{a} - \frac{R}{8a^3}\right)\right\} (c - R\cos\chi)\, R \, dR \, d\chi \, d\phi$$

$$+ \pi a^2 \iiint \Sigma\, \beta_n \left\{ \frac{1}{n}\frac{R^n}{a^n}\cos n\chi + \frac{\sigma}{4}\left[\left(\frac{1}{n}\frac{R^{n+1}}{a^{n+1}} - \frac{1}{n-1}\frac{R^{n-1}}{a^{n-1}}\right)\cos(n-1)\chi \right.\right.$$

$$\left.\left. + \frac{1}{n(n+1)}\frac{R^{n+1}}{a^{n+1}}\cos(n+1)\chi\right]\right\} (c - R\cos\chi)\, R\, dR\, d\chi\, d\phi$$

$$+ \pi a^2 \iiint \left\{ (L-1)\,\Sigma\,\frac{\beta_n^2}{2} - \frac{2L+1}{4}\sigma\,\Sigma\,\beta_n\beta_{n+1} - \frac{L-1}{16}\sigma^2\,\Sigma\,\beta_n^2 \right.$$

$$\left. + \frac{L-1}{4}\sigma\,\Sigma\,\beta_n^2\left(\frac{R}{a}\cos\chi + \frac{\sigma}{4}\frac{R^2}{a^2}\right)\right\} (c - R\cos\chi)\, R\, dR\, d\chi\, d\phi$$

$$= 2\pi^2 a^2 \int_0^{2\pi} \left[c\left\{(L + \tfrac{1}{2})\frac{\rho^2}{2} - \frac{\rho^4}{8a^2}\right\} - \cos\chi\left\{(L+\tfrac{1}{2})\frac{\rho^3}{3} - \frac{\rho^5}{10a^2}\right\}\right.$$

$$\left. + c\sigma\cos\chi\left\{\frac{L-1}{2}\frac{\rho^3}{3a} - \frac{\rho^5}{40a^3}\right\}\right] d\chi$$

$$+ 2\pi^2 a^4 c \int_0^{2\pi} \Sigma\,\beta_n \cos n\chi\, \Sigma \left\{ \frac{\beta_n}{n}\cos n\chi - \frac{\sigma\beta_n}{4n(n-1)}\cos(n-1)\chi \right.$$

$$\left. + \frac{\sigma\beta_n}{4n(n+1)}\cos(n+1)\chi\right\}(1 - \sigma\cos\chi)\, d\chi$$

$$+ 2\pi^3 a^4 c \left\{ (L-1)\,\Sigma\,\frac{\beta_n^2}{2} - \frac{2L+1}{4}\sigma\,\Sigma\,\beta_n\beta_{n+1}\right\}.$$

In the first integral, substitute

$$\rho^p = a^p \left\{ 1 + p\,\Sigma\,\beta_n \cos n\chi + \frac{p(p-1)}{2}\,\Sigma\left(\frac{\beta_n^2}{2} + \cos\chi\,\Sigma\,\beta_n\beta_{n+1} + \ldots\right)\right\}.$$

On integration this gives

$$2\pi^3 a^4 c \left\{ L + \tfrac{1}{4} + (L-1)\,\Sigma\,\frac{\beta_n^2}{2} - \frac{2L+1}{4}\sigma\,\Sigma\,\beta_n\beta_{n+1}\right\}.$$

The second integral may be written in the form

$$2\pi^2 a^4 c \int_0^{2\pi} \Sigma\,\beta_n \cos n\chi\, \Sigma \left(\frac{\beta_n}{n} - \frac{\sigma\beta_{n+1}}{4n(n+1)} + \frac{\sigma\beta_{n-1}}{4n(n-1)}\right)\cos n\chi\,(1 - \sigma\cos\chi)\, d\chi,$$

giving on integration

$$2\pi^3 a^4 c \left\{ \Sigma \frac{\beta_n^2}{n} - \sigma \Sigma \frac{(2n+1)\beta_n \beta_{n+1}}{2n(n+1)} \right\}.$$

Therefore, adding the three integrals together,

$$U = 2\pi^3 a^4 c \left\{ L + \tfrac{1}{4} + \Sigma \left(L - 1 + \frac{1}{n} \right) \beta_n^2 - \sigma \Sigma \left[\frac{2L+1}{2} + \frac{2n+1}{2n(n+1)} \right] \beta_n \beta_{n+1} \right\} \quad (16).$$

Let a_0 be the mean radius of the cross-section.
Then

$$\pi a_0^2 = \pi a^2 \left(1 + \Sigma \frac{\beta_n^2}{2} \right) \quad \ldots \ldots \ldots \quad (17).$$

Let $2\pi^2 a_0^2 c_0$ be the volume of the ring.
Then

$$2\pi^2 a_0^2 c_0 = 2\pi^2 a^2 c \left(1 + \Sigma \frac{\beta_n^2}{2} - \sigma \Sigma \beta_n \beta_{n+1} \right) \quad \ldots \ldots \quad (18).$$

Substituting, we obtain

$$U = 2\pi^3 a_0^4 c_0 \left\{ \log \frac{8c_0}{a_0} + \tfrac{1}{4} - \Sigma \left[\beta_n^2 \frac{n-1}{n} - \sigma \beta_n \beta_{n+1} \frac{(n-1)(3n+1)}{4n(n+1)} \right] \right\}.$$

M. POINCARÉ gives

$$U = 2\pi^3 a_0^4 c_0 \left\{ \log \frac{8c_0}{a_0} + \tfrac{1}{4} - \Sigma \frac{n-1}{n} \beta_n^2 \right\}.$$

(TISSERAND, 'Méc. Cél.,' vol. 2, p. 166.)

[This is correct to the first power of σ. The term in $\sigma^2 \beta_2$ is important, for in the equilibrium position of the ring β_2 is of order σ^2, and therefore this term is of the same order as the term in β_2^2. By (10) and (13), the more correct value of U is

$$2\pi^3 a_0^4 c_0 \left\{ L + \tfrac{1}{4} - \frac{L - \frac{2}{3}}{8} \sigma^2 - \frac{3(L - \frac{19}{12})}{512} \sigma^4 - \beta_2 \frac{\sigma^2}{8} (L - \tfrac{5}{12}) \right.$$
$$\left. - \Sigma \left[\beta_n^2 \frac{n-1}{n} - \sigma \beta_n \beta_{n+1} \frac{(n-1)(3n+1)}{4n(n+1)} \right] \right\} \quad . \quad . \quad (19).$$

July 22, 1893.]*

* This was inserted in consequence of an observation of one of the Referees, who noticed, what I had overlooked, that the less exact value of U failed in § 14 to give the correct equilibrium value of β_2.

§ 14. To find the fluted oscillations, it is necessary to determine the kinetic energy in the disturbed motion.

Let Φ be the velocity potential of this motion.

Since the bounding surface is given by

$$\mathrm{R} = a\{1 + \Sigma\beta_n \cos n\chi\},$$

therefore

$$\frac{\partial\Phi}{\partial\mathrm{R}} + \frac{\partial\mathrm{R}}{\partial c}\dot{c} = \dot{a}(1+\Sigma\,\beta_n\cos n\chi) + a\,\Sigma\,\dot{\beta}_n\cos n\chi - a\,\Sigma\,(n\beta_n\sin n\chi)\left(\frac{1}{\mathrm{R}^2}\frac{\partial\Phi}{\partial\chi} + \frac{\partial\chi}{\partial c}\dot{c}\right)$$

at the surface of the ring.

Now

$$\frac{\partial\mathrm{R}}{\partial c} = \cos\chi \quad \text{and} \quad \frac{\partial\chi}{\partial c} = -\frac{\sin\chi}{\mathrm{R}}.$$

Therefore

$$\frac{\partial\Phi}{\partial\mathrm{R}} = \dot{a} - \dot{c}\cos\chi + \Sigma\,(a\dot{\beta}_n + \dot{a}\beta_n)\cos n\chi - a\,\Sigma\,(n\beta_n\sin n\chi)\left(\frac{1}{\mathrm{R}^2}\frac{\partial\Phi}{\partial\chi} - \frac{\sin\chi}{\mathrm{R}}\dot{c}\right).$$

Since

$$a^2c\left(1 + \Sigma\frac{\beta_n^2}{2}\right) = \text{const.},$$

$$\frac{\dot{c}}{c} + \frac{2\dot{a}}{a} + \Sigma\beta_n\dot{\beta}_n = 0.$$

Therefore

$$\frac{\partial\Phi}{\partial\mathrm{R}} = -\dot{c}\left(\cos\chi + \frac{a}{2c}\right) - \frac{a}{2}\Sigma\beta_n\dot{\beta}_n + \Sigma\,(a\dot{\beta}_n + \dot{a}\beta_n)\cos n\chi$$

$$- a\,\Sigma\,(n\beta_n\sin n\chi)\left(\frac{1}{\mathrm{R}^2}\frac{\partial\Phi}{\partial\chi} - \frac{\sin\chi}{\mathrm{R}}\dot{c}\right).$$

Approximately

$$\frac{\partial\Phi}{\partial\mathrm{R}} = -\dot{c}\left(\cos\chi + \frac{a}{2c}\right).$$

Therefore

$$\Phi = -a\dot{c}\left\{\frac{\mathrm{R}}{a}\cos\chi + \frac{a}{4c}\frac{\mathrm{R}^2}{a^2}\right\}.$$

This gives

$$\frac{\partial\Phi}{\partial\chi} = a\dot{c}\frac{\mathrm{R}}{a}\sin\chi.$$

Substituting in the last term, we see that it vanishes: the second term may also be omitted, giving

$$\frac{\partial\Phi}{\partial\mathrm{R}} = -\dot{c}\left(\cos\chi + \frac{a}{2c}\right) + \Sigma\,(a\dot{\beta}_n + \dot{a}\beta_n)\cos n\chi$$

at the surface.

Therefore

$$\Phi = -\dot{c}a\left(\frac{R}{a}\cos\chi + \frac{a}{4c}\frac{R^2}{a^2}\right) + \frac{1}{n}\Sigma\left(a\dot{\beta}_n + \dot{a}\beta_n\right)a\frac{R^n}{a^n}\cos n\chi.$$

The term in $\dot{a}\beta_n$ may be omitted.

It is easily seen, by § 4, that a more approximate solution is obtained by assuming

$$\Phi = -a\dot{c}A_1\left\{\frac{R}{a}\cos\chi + \frac{a}{4c}\frac{R^2}{a^2} + \frac{a}{32c}\frac{R^3}{a^3}\cos\chi\right\}$$

$$+\Sigma A_n\left\{\frac{R^n}{a^n}\cos n\chi + \frac{a}{4c}\frac{R^{n+1}}{a^{n+1}}\cos(n-1)\chi\right\} \quad . \quad . \quad . \quad . \quad (20).$$

The constants are easily found to be

$$A_1 = 1 - \frac{3\sigma^2}{32},$$

$$A_n = a^2\left(\frac{\dot{\beta}_n}{n} - \frac{(n+2)}{4n(n+1)}\sigma\dot{\beta}_{n+1}\right).$$

At the surface of the ring, therefore,

$$\Phi = -a\dot{c}\left\{\cos\chi\left(1 - \frac{\sigma^2}{16}\right) + \frac{\sigma}{4}\right\} + a^2\Sigma\left(\frac{\dot{\beta}_n}{n} - \frac{\sigma(n+2)}{4n(n+1)}\dot{\beta}_{n+1}\right)\left(\cos n\chi + \frac{\sigma}{4}\cos(n-1)\chi\right)$$

$$= -a\dot{c}\left\{\cos\chi\left(1 - \frac{\sigma^2}{16}\right) + \frac{\sigma}{4}\right\} + a^2\Sigma\left(\frac{\dot{\beta}_n}{n} - \frac{\sigma\dot{\beta}_{n+1}}{2n(n+1)}\right)\cos n\chi$$

and

$$\frac{d\Phi}{dn} = -\dot{c}\left(\cos\chi + \frac{\sigma}{2}\right) + a\,\Sigma\,\dot{\beta}_n\cos n\chi.$$

The kinetic energy is given by

$$2T = \iint \Phi\frac{d\Phi}{dn}dS$$

$$= 2\pi ac\int\Phi\frac{d\Phi}{dR}(1 - \sigma\cos\chi)\,d\chi$$

$$= 2\pi^2a^2c\left\{\dot{c}^2\left(1 - \frac{9\sigma^2}{16}\right) + a^2\Sigma\left(\frac{\dot{\beta}_n^2}{n} - \frac{\sigma\dot{\beta}_n\dot{\beta}_{n+1}}{n}\right)\right\}. \quad . \quad . \quad . \quad (21).$$

To this must be added the kinetic energy of the undisturbed motion, which is given by

$$2\mathrm{T} = \pi\dot{\phi}^2 \int_0^{2\pi} \int_0^{\rho} (c - \mathrm{R}\cos\chi)\, 3\mathrm{R}\, d\mathrm{R}\, d\chi$$
$$= \mathrm{M}c^2\dot{\phi}^2 (1 + \tfrac{3}{4}\sigma^2 + \tfrac{3}{2}\sigma^2\beta_2) \quad . \quad . \quad . \quad (22),$$

where $\dot{\phi}$ is the undisturbed angular velocity of the fluid.

Therefore

$$2\mathrm{T} = \mathrm{M}\left\{\dot{c}^2\left(1 - \frac{9\sigma^2}{16}\right) + a^2\Sigma\frac{\dot{\beta}_n^2 - \sigma\dot{\beta}_n\dot{\beta}_{n+1}}{n} + c^2\dot{\phi}^2(1 + \tfrac{3}{4}\sigma^2 + \tfrac{3}{2}\sigma^2\beta_2)\right\} \quad . \quad (23).$$

§ 15. Also

$$\mathrm{U} = \gamma\frac{\mathrm{M}^2}{2\pi c}\left\{\log\frac{8c}{a} + \tfrac{1}{4} - \&\mathrm{c.} - \frac{\beta_2\sigma^2}{8}\left(\log\frac{8c}{a} - \tfrac{5}{12}\right)\right.$$
$$\left. - \Sigma\left(\frac{n-1}{n}\beta_n^2 - \frac{(3n+1)(n-1)}{2n(n+1)}\sigma\beta_n\beta_{n+1}\right)\right\} \quad . \quad . \quad (24),$$

where a and c now denote the mean radii of the cross-section and of the ring respectively; $\sigma = a/c$, and is small; γ is the constant of gravitation.

Retain only the terms of the highest order; LAGRANGE's equations give

$$\mathrm{M}a^2\frac{\ddot{\beta}_n}{n} + \frac{\mathrm{M}^2}{2\pi c}\gamma\frac{n-1}{2n}\beta_n = 0.$$

Thus the period of the oscillation of the type $\beta_n \cos n\chi$ is given approximately by

$$\ddot{\beta}_n + \pi\frac{(n-1)}{2}\gamma\beta_n = 0. \quad . \quad . \quad . \quad (25).$$

The time of a complete oscillation is $\sqrt{\dfrac{8\pi}{(n-1)\gamma}}$, and the time a fluted wave of this type would take to travel round the ring is

$$n\sqrt{\frac{8\pi}{(n-1)\gamma}}.$$

[The equation giving the period of the disturbance of type $\beta_2 \cos 2\chi$ is

$$\mathrm{M}\left\{a^2\frac{\ddot{\beta}_2}{2} - \tfrac{3}{4}c^2\sigma^2\dot{\phi}^2\right\} + \gamma\frac{\mathrm{M}^2}{2\pi c}\left\{\beta_2 + \frac{\sigma^2}{8}\left(\log\frac{8c}{a} - \tfrac{5}{12}\right)\right\} = 0.$$

Therefore the value of β_2 in the undisturbed motion is given by

$$\begin{aligned}\beta_2 &= \tfrac{3}{4}\,\pi\gamma\dot{\phi}^2 - \frac{\sigma^2}{8}\left(\log\frac{8c}{a} - \tfrac{5}{12}\right)\\ &= \frac{3a^2}{4c^2}\left(\log\frac{8c}{a} - \tfrac{5}{4}\right) - \frac{a^2}{8c^2}\left(\log\frac{8c}{a} - \tfrac{5}{12}\right), \text{ by (26)},\\ &= \tfrac{5}{8}\,\frac{a^2}{c^2}\left(\log\frac{8c}{a} - \tfrac{17}{12}\right),\end{aligned}$$

agreeing with the result given *supra*, p. 94.—July 22, 1893.]*

An interesting oscillation of the ring is that in which the mean radius increases and diminishes, necessitating a decrease and increase of the cross-section.

We have

$$\left.\begin{aligned} Mc^2\dot{\phi} = \text{const.}\\ M\ddot{c} - Mc\dot{\phi}^2 - \gamma\,\frac{M^2}{2\pi}\,\frac{d}{dc}\left(\frac{\log\frac{8c}{a} + \frac{1}{4}}{c}\right) = 0\end{aligned}\right\}$$

Remembering that a^2c is constant,

$$\begin{aligned}\frac{d}{dc}\,\frac{\log\frac{8c}{a} + \frac{1}{4}}{c} &= -\frac{1}{c^2}\left(\log\frac{8c}{a} + \tfrac{1}{4}\right) + \frac{1}{c}\left(\frac{1}{c} + \frac{1}{a}\,\frac{a}{2c}\right)\\ &= -\frac{1}{c^2}\left(\log\frac{8c}{a} - \tfrac{5}{4}\right).\end{aligned}$$

Therefore,

$$M\ddot{c} - Mc\dot{\phi}^2 + \frac{\gamma M^2}{2\pi c^2}\left(\log\frac{8c}{a} - \tfrac{5}{4}\right) = 0.$$

Thus the angular velocity in the position of relative equilibrium is given by

$$\begin{aligned}\dot{\phi}^2 &= \frac{\gamma M^2}{2\pi M c^3}\left(\log\frac{8c}{a} - \tfrac{5}{4}\right)\\ &= \gamma\pi\,\frac{a^2}{c^2}\left(\log\frac{8c}{a} - \tfrac{5}{4}\right) \quad \ldots\ldots\ldots \quad (26),\end{aligned}$$

agreeing with the results found, *ante*, p. 94.

For the small oscillation, write $c = c_0 + x$, $a = a_0 + y$, in the above equation.

Then,

$$a_0^2 x + 2\,a_0 c_0 y = 0, \quad \text{or} \quad y = -\tfrac{1}{2}\,\frac{a_0}{c_0}\,x.$$

* See note, p. 1060.

Therefore

$$\mathrm{M}\ddot{x} - \frac{\mathrm{M}c^4\dot{\phi}^2}{(c_0 + x)^3} + \frac{\gamma\,\mathrm{M}^2}{2\pi\,(c_0 + x)^2}\left(\log\frac{8\,(c_0 + x)}{(a_0 + y)} - \tfrac{5}{4}\right) = 0.$$

Therefore

$$\ddot{x} - \gamma\,\frac{a_0^2 c_0^2\left(\log\frac{8c_0}{a_0} - \frac{5}{4}\right)}{(c_0 + x)^3} + \frac{\gamma\,2\pi\,a_0^2 c_0}{2\pi\,(c_0 + x)^2}\left(\log\frac{8\,(c_0 + x)}{a_0 + y} - \tfrac{5}{4}\right) = 0,$$

or

$$\ddot{x} + 3\gamma\,\frac{a_0^2}{c_0^2}\left(\log\frac{8c_0}{a_0} - \tfrac{5}{4}\right)x - 2\gamma\,\frac{a_0^2}{c_0^2}\left(\log\frac{8c_0}{a_0} - \tfrac{5}{4}\right)x + \tfrac{3}{2}\,\gamma\,\frac{a_0^2}{c_0^2}\,x = 0,$$

or

$$\ddot{x} + \gamma\,\frac{a_0^2}{c_0^2}\left(\log\frac{8c_0}{a_0} + \tfrac{1}{4}\right)x = 0, \quad . \; . \; . \; . \; . \; . \; . \; . \quad (27),$$

giving for the time of a complete oscillation

$$\frac{2\pi\,c_0}{a_0}\Big/\sqrt{\left\{\gamma\left(\log\frac{8c_0}{a_0} + \tfrac{1}{4}\right)\right\}}.$$

The oscillations might be found more approximately, but this hardly seems worth while, as the ring will be proved unstable for disturbances of a different kind.

§ 16. *The effect of long beaded waves.*

As before, let U be the exhaustion of potential energy of the ring in its disturbed state. In this case the ring is disturbed so that its central circle remains an exact circle of radius c, the cross-section is always a circle, but the radius of this circle varies with the azimuth. Let it be given by

$$\rho = a\{1 + \Sigma\,(\alpha_n \sin n\phi + \beta_n \cos n\phi)\}.$$

Let V_i be the potential inside; V_0, outside.

Then

$$\mathrm{U} = \tfrac{1}{2}\int \mathrm{V}_i\,dm$$

$$= \tfrac{1}{2}\int_0^{2\pi}\!\!\int_0^{\rho}\!\!\int_0^{2\pi} \mathrm{V}_i\,(c - \mathrm{R}\cos\chi)\,\mathrm{R}\,d\mathrm{R}\,d\chi\,d\phi.$$

If V_i be expressed in the form

$$v_0 + v_1 \cos\chi + v_2 \cos 2\chi + \ldots,$$

the only terms we need are v_0 and v_1,

Also, the only terms of the second order in the small quantities α and β, which need be retained are those independent of ϕ, as the others vanish on integration.

§ 17. To determine V_i we must first find V_0.

Let ϖ', z', ϕ' be the coordinates of any external point, ϖ, z, ϕ of any internal point. Then

$$V_0 = \iiint \frac{\varpi\, d\varpi\, dz\, d\phi}{\sqrt{\{\varpi'^2 + \varpi^2 - 2\varpi\varpi' \cos(\phi' - \phi) + (z' - z)^2\}}}.$$

Let $\varpi = c - x$.

Then

$$V_0 = \iiint \frac{(c - x)\, dx\, dz\, d\phi}{\sqrt{\{(c - x)^2 + \varpi'^2 - 2(c - x)\varpi' \cos(\phi' - \phi) + (z' - z)^2\}}}$$

$$= \iiint e^{-x(d/dc) - z(d/dz')}\, dx\, dz \frac{c\, d\phi}{\sqrt{\{c^2 + \varpi'^2 2c\varpi' \cos(\phi' - \phi) + z'^2\}}}$$

$$= \int_0^{2\pi} \iint \left\{1 - \left(x\frac{d}{dc} + z\frac{d}{dz'}\right) + \frac{1}{2!}\left(x\frac{d}{dc} + z\frac{d}{dz'}\right)^2 \ldots\right\} dx\, dz \frac{c\, d\phi}{\sqrt{\{\varpi'^2 + c^2 - 2c\varpi' \cos(\phi' - \phi) + z'^2\}}}$$

where the double integral is taken over a circle of radius ρ.

Therefore

$$V_0 = \int_0^{2\pi} \left\{\pi\rho^2 + \tfrac{1}{8}\pi\rho^4\left(\frac{d^2}{dc^2} + \frac{d^2}{dz'^2}\right) + \tfrac{1}{192}\pi\rho^6\left(\frac{d^2}{dc^2} + \frac{d^2}{dz'^2}\right)^2 + \ldots\right\} \frac{c\, d\phi}{\sqrt{\{\varpi'^2 + c^2 - 2c\varpi' \cos(\phi' - \phi) + z'^2\}}}.$$

Now

$$\rho^p = a^p \left\{1 + p\Sigma(\alpha_n \sin n\phi + \beta_n \cos n\phi) + \frac{p(p-1)}{2}\Sigma\left[\frac{\alpha_n^2 + \beta_n^2}{2} + (\alpha_n\alpha_{n+1} + \beta_n\beta_{n+1})\cos\phi + \ldots\right]\right\}.$$

Substituting this, and writing $\dfrac{d^2}{dc^2} + \dfrac{d^2}{dz'^2} = \nabla^2$

$$V_0 = \pi a^2 \int_0^{2\pi} \left\{1 + \frac{a^2}{8}\nabla^2 + \frac{a^4}{192}\nabla^4\right\} \frac{c\, d\phi}{\sqrt{\{\varpi'^2 + c^2 - 2c\varpi' \cos(\phi' - \phi) + z'^2\}}}$$

$$+ \pi a^2 \int_0^{2\pi} \left\{2 + \frac{a^2}{2}\nabla^2 + \frac{a^4}{32}\nabla^4\right\} \frac{c\Sigma(\alpha_n \sin n\phi + \beta_n \cos n\phi)\, d\phi}{\sqrt{\{\varpi'^2 + c^2 - 2c\varpi' \cos(\phi' - \phi) + z'^2\}}}$$

$$+ \pi a^2 \int_0^{2\pi} \Sigma\frac{\alpha_n^2 + \beta_n^2}{2}\left\{1 + \frac{3a^2}{4}\nabla^2 + \frac{5a^4}{64}\nabla^4\right\} \frac{c\, d\phi}{\sqrt{\{\varpi'^2 + c^2 - 2c\varpi' \cos(\phi' - \phi) + z'^2\}}}$$

$$= \pi a^2 \int_0^{2\pi} \left\{ 1 + \frac{a^2}{8} \nabla^2 + \frac{a^4}{192} \nabla^4 + \ldots \right\} \frac{c\, d\phi}{\sqrt{(\varpi'^2 + c^2 - 2c\varpi' \cos\phi + z'^2)}}$$

$$+ \pi a^2 \Sigma (\alpha_n \sin n\, \phi' + \beta_n \cos n\, \phi') \int_0^{2\pi} \left\{ 2 + \frac{a^2}{2} \nabla^2 \right\} \frac{c \cos n\phi\, d\phi}{\sqrt{(\varpi'^2 + c^2 - 2c\varpi' \cos\phi + z'^2)}}$$

$$+ \pi a^2 \Sigma \frac{\alpha_n^2 + \beta_n^2}{2} \int_0^{2\pi} \left\{ 1 + \frac{3a^2}{4} \nabla^2 + \ldots \right\} \frac{c\, d\phi}{\sqrt{\{\varpi'^2 + c^2 - 2c\varpi' \cos\phi + z'^2\}}} \quad . \quad . \quad (28).$$

§ 18. These integrals admit of simplification.
Denoting

$$\int_0^{2\pi} \frac{\cos n\phi\, d\phi}{\sqrt{\{c^2 + \varpi'^2 - 2c\varpi' \cos\phi + z'^2\}}} \text{ by I}$$

$$\frac{d^2 \mathrm{I}}{d\varpi'^2} + \frac{1}{\varpi'} \frac{d\mathrm{I}}{d\varpi'} + \frac{d^2\mathrm{I}}{dz'^2} - \frac{n^2}{\varpi'^2} \mathrm{I} = 0,$$

since I cos $n\ \phi$ is a solution of LAPLACE's equation.

Symmetry shows that

$$\frac{d^2\mathrm{I}}{dc^2} + \frac{1}{c}\frac{d\mathrm{I}}{dc} + \frac{d^2\mathrm{I}}{dz'^2} - \frac{n^2}{c^2}\mathrm{I} = 0,$$

therefore

$$\left(\frac{d^2}{dc^2} + \frac{d^2}{dz'^2}\right) \mathrm{I}c = \frac{d\mathrm{I}}{dc} + \frac{n^2}{c}\mathrm{I},$$

and

$$\left(\frac{d^2}{dc} + \frac{d^2}{dz'^2}\right)^2 \mathrm{I}c = -\frac{d}{dc}\left(\frac{1}{c}\frac{d\mathrm{I}}{dc}\right) - 2\frac{n^2}{c^2}\frac{d\mathrm{I}}{dc} + \frac{n^4}{c^3}\mathrm{I}.$$

Substituting, we obtain

$$\mathrm{V}_0 = 2\pi a^2 c \left\{ 1 + \tfrac{1}{8}\frac{a^2}{cdc} - \tfrac{1}{192}\left(\frac{a^2}{cdc}\right)^2 + \ldots \right\} \int_0^{\pi} \frac{d\phi}{\sqrt{(c^2 + \varpi'^2 - 2c\varpi' \cos\phi + z'^2)}}$$

$$+ 4\pi a c\, \Sigma (\alpha_n \sin n\phi' + \beta_n \cos n\phi') \left\{ 1 + \frac{a^2 n^2}{4c^2} + \frac{a^4 n^4}{64 c^4} + \left(\tfrac{1}{4} - \frac{a^2 n^2}{32 c^2}\right)\frac{a^2}{cdc} \right.$$

$$\left. - \tfrac{1}{64}\left(\frac{a^2}{cdc}\right)^2 \right\} \int_0^{\pi} \frac{\cos n\phi\, d\phi}{\sqrt{(c^2 + \varpi'^2 - 2c\varpi' \cos\phi + z'^2)}}$$

$$+ 2\pi a^2 c\, \Sigma \frac{\alpha_n^2 + \beta_n^2}{2} \left\{ 1 + \tfrac{3}{4}\frac{a^2}{cdc} - \tfrac{5}{32}\left(\frac{a^2}{cdc}\right)^2 \right\} \int_0^{\pi} \frac{d\phi}{\sqrt{(c^2 + \varpi'^2 - 2c\varpi' \cos\phi + z'^2)}} \quad (29).$$

It is important to notice that at the surface of the ring differentiation with respect to c lowers the order of the terms once in a/c. Therefore the operation $\frac{a^2}{c} \cdot \frac{d}{dc}$ raises the order once in a/c. Provided that n is not greater than $\sqrt{(c/a)}$, that is, when the waves are long, the value of V_0 is given by § 17 correctly to the order $(a/c)^2$.

We shall, however, simplify the work by rejecting all but the most important terms, though it would be quite easy to retain terms of higher orders.

§ 19. Let the point ϖ', z', ϕ' be near the ring, and with the notation used before be the point $R.\chi$.

The most important term in $\int_0^\pi \frac{d\phi}{\sqrt{(\varpi'^2 - 2\varpi' c \cos\phi + c^2 + z'^2)}}$ is $\log \frac{8c}{R}$.

The integral $\int_0^\pi \frac{\cos n\phi \, d\phi}{\sqrt{(\varpi'^2 - 2\varpi' c \cos\phi + c^2 + z'^2)}}$, being of the form $\frac{1}{\sqrt{(2\varpi' c)}} \int_0^\pi \frac{\cos n\phi \, d\phi}{\sqrt{(q - \cos\phi)}}$ is

$$\frac{1}{\sqrt{(2\varpi' c)}} \frac{1}{\sqrt{2}} \log \frac{16(q+1)}{q-1} - \frac{2\sqrt{2}}{\sqrt{(2\varpi' c)}} \left(1 + \tfrac{1}{3} + \ldots + \frac{1}{2n-1}\right)$$

(J. J. THOMSON, 'Motion of Vortex-rings,' p. 26)

$$= \log \frac{8c}{R} - 2f(n),$$

where

$$f(n) = 1 + \tfrac{1}{3} + \ldots + \frac{1}{2n-1}.$$

Writing ϕ for the azimuth, instead of ϕ', we have, therefore, that near the ring outside

$$V_0 = 2\pi a^2 \left\{1 + \Sigma \frac{\alpha_n^2 + \beta_n^2}{2}\right\} \log \frac{8c}{R}$$
$$+ 4\pi a^2 \Sigma (\alpha_n \sin n\phi + \beta_n \cos n\phi) \left\{\log \frac{8c}{R} - 2f(n)\right\} \quad . \quad . \quad (30).$$

Inside the ring, therefore,

$$V_i = 2\pi a^2 \left\{\log \frac{8c}{a} + \tfrac{1}{2}\left(1 - \frac{R^2}{a^2}\right)\right\}$$
$$+ 4\pi a^2 \Sigma (\alpha_n \sin n\phi + \beta_n \cos n\phi) \left\{\log \frac{8c}{a} - 2f(n)\right\}$$
$$+ C. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (31),$$

where C is a constant of the second order in α_n, β_n, &c.

As in § 11, we find that at the surface of the ring,

$$V_0 - V_i = 2\pi a^2 \Sigma \frac{\alpha_n^2 + \beta_n^2}{2} \log \frac{8c}{a} + \frac{a^2}{2} \Sigma (\alpha_n \sin n\phi + \beta_n \cos n\phi)^2 4\pi$$
$$+ a \Sigma (\alpha_n \sin n\phi + \beta_n \cos n\phi) [- 4\pi a \Sigma (\alpha_n \sin n\phi + \beta_n \cos n\phi)] - C.$$

Since V_0 and V_i are equal at the surface,

$$C = 2\pi a^2 \left(\log \frac{8c}{a} - 1\right) \Sigma \frac{\alpha_n^2 + \beta_n^2}{2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (32).$$

Now

$$U = \tfrac{1}{2} \int V_i \, dm$$

$$= \pi a^2 \int_0^{2\pi} \int_0^{\rho} \int_0^{2\pi} \left\{ \log \frac{8c}{a} + \tfrac{1}{2}\left(1 - \frac{R^2}{a^2}\right) + 2 \Sigma \left(\log \frac{8c}{a} - 2f(n)\right) (\alpha_n \sin n\phi + \beta_n \cos n\phi) \right.$$
$$\left. + \left(\log \frac{8c}{a} - 1\right) \Sigma \frac{\alpha_n^2 + \beta_n^2}{2} \right\} (c - R \cos \chi) R \, dR \, d\chi \, d\phi$$

$$= 2\pi^2 a^2 c \int_0^{2\pi} \left\{ \left(\log \frac{8c}{a} + \tfrac{1}{2}\right) \frac{\rho^2}{2} - \frac{\rho^4}{8a^2} + 2 \Sigma \left(\log \frac{8c}{a} - 2f(n)\right) (\alpha_n \sin n\phi + \beta_n \cos n\phi) \frac{\rho^2}{2} \right.$$
$$\left. + \left(\log \frac{8c}{a} - 1\right) \Sigma \frac{\alpha_n^2 + \beta_n^2}{2} \frac{\rho^2}{2} \right\} d\phi$$

$$= 2\pi^3 a^4 c \left\{ \log \frac{8c}{a} + \tfrac{1}{4} + \left(\log \frac{8c}{a} - 1\right) \Sigma (\alpha_n^2 + \beta_n^2) \right.$$
$$\left. + \left(2 \log \frac{8c}{a} - 4f(n)\right) \Sigma (\alpha_n^2 + \beta_n^2) \right\}.$$

Let a_0 be the mean value of a.
Then

$$U = 2\pi^3 a_0^4 c \left\{ \log \frac{8c}{a_0} + \tfrac{1}{4} + 2 \Sigma \left(\log \frac{8c}{a_0} - 2f(n) - \tfrac{1}{4}\right) (\alpha_n^2 + \beta_n^2) \right\} \quad . \quad (33).$$

§ 20. When the ring is thin and n is small the change in the radius of the cross-section is slow, and an approximation to the disturbed motion is easily found by the method of parallel sections.

$$\pi \rho^2 v = \text{const.} = C$$

$$2T = \int_0^{2\pi} \pi \rho^2 \, c \, d\phi \, v^2$$

$$h = \int_0^{2\pi} \pi \rho^2 \, c \, d\phi \, cv = 2\pi c^2 C.$$

Therefore

$$2T = \int_0^{2\pi} \frac{cC^2}{\pi\rho^2}\, d\phi.$$
$$= \frac{2cC^2}{a^2}\{1 + \tfrac{3}{2}\Sigma(\alpha_n^2 + \beta_n^2)\}$$
$$= \frac{2cC^2}{a_0^2}\{1 + 2\Sigma(\alpha_n^2 + \beta_n^2)\}$$
$$= \frac{h^2}{Mc^2}\{1 + 2\Sigma(\alpha_n^2 + \beta_n^2)\}.$$

Let A_n be the component of momentum corresponding to the coordinate α_n. Then the whole kinetic may be expressed in the form

$$\frac{h^2}{2Mc^2}\{1 + 2\Sigma(\alpha_n^2 + \beta_n^2)\} + \tfrac{1}{2}\Sigma k_n A_n^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (34),$$

where the terms in the summation are necessarily positive. Thus

$$T - U = \tfrac{1}{2}\Sigma k_n A_n^2 + \frac{h^2}{2Mc^2}\{1 + 2\Sigma(\alpha_n^2 + \beta_n^2)\}$$
$$- \frac{2M^2}{\pi c}\{L + \tfrac{1}{4} + 2\Sigma[L - 2f(n) - \tfrac{1}{4}](\alpha_n^2 + \beta_n^2)\} \quad . \quad . \quad (35).$$

HAMILTON'S equations will give

$$\frac{d}{dt}(k_n A_n) + \frac{2h^2}{Mc^2}\alpha_n - \frac{2M^2}{\pi c}[L - 2f(n) - \tfrac{1}{4}]\alpha_n = 0 \quad . \quad . \quad . \quad . \quad (36).$$

The steady motion will be stable or unstable as

$$\frac{2h^2}{Mc^2} - \frac{2M^2}{\pi c}[L - 2f(n) - \tfrac{1}{4}] \text{ is } > \text{ or } < 0.$$

That is, according as

$$-[L - 4f(n) + \tfrac{3}{4}] \text{ is } > \text{ or } < 0.$$

Now, when $n = 1$

$$L - 4f(n) + \tfrac{3}{4} = \log\frac{8c}{a_0} - 3\tfrac{1}{4}.$$

When $c > 3a_0$, this is positive: therefore for any ring whose cross-section is smaller than this, beaded waves cause instability.

§ 21. Next consider a disturbance which leaves the cross-section of the ring by any plane through the axis an exact circle of radius a, but the centre of the section is

displaced to the point whose cylindrical coordinates are $c - \xi$, ζ, ϕ,
where

$$\xi = c\,\Sigma\,(\alpha_n \sin n\phi + \beta_n \cos n\phi),$$

and

$$\zeta = c\,\Sigma\,(\gamma_n \sin n\phi + \delta_n \cos n\phi),$$

ξ and ζ are taken to be small quantities compared with a.

Let $c - \xi - x$, $z + \zeta$, ϕ be coordinates of any point of the ring, and ϖ', z', ϕ' of any external point.

The potential of the ring at ϖ', z'. ϕ' is given by

$$\begin{aligned}
\mathrm{V}_0 &= \iiint \frac{(c - x - \xi)\,dx\,dz\,d\phi}{\sqrt{\{\varpi'^2 - 2(c - x - \xi)\,\varpi' \cos(\phi' - \phi) + (c - x - \xi)^2 + (z' - z - \zeta)^2\}}} \\
&= \iiint e^{-(x+\xi)\,d/dc - (z+\zeta)\,d/dz'}\,dx\,dz\,\frac{c\,d\phi}{\sqrt{\{\varpi'^2 + c^2 - 2c\varpi' \cos(\phi' - \phi) + z'^2\}}} \\
&= \int_0^{2\pi} e^{-\xi(d/dc) - \zeta(d/dz')} \iint \left\{1 - x\frac{d}{dc} - z\frac{d}{dz'}\right. \\
&\qquad\qquad \left. + \ldots\right\} dx\,dz\,\frac{c\,d\phi}{\sqrt{\{\varpi'^2 + c^2 - 2c\varpi' \cos(\phi' - \phi) + z'^2\}}} \\
&= \int_0^{2\pi} e^{-\xi(d/dc) - \zeta(d/dz')}\,\pi a^2 \left\{1 + \frac{a^2}{8}\left(\frac{d^2}{dc^2} + \frac{d^2}{dz'^2}\right)\right. \\
&\qquad\qquad \left. + \ldots\right\} \frac{c\,d\phi}{\sqrt{\{\varpi'^2 + c^2 - 2c\varpi' \cos(\phi' - \phi) + z'^2\}}} \\
&= \pi a^2 \left\{1 + \frac{a^2}{8}\left(\frac{d^2}{dc^2} + \frac{d^2}{dz'^2}\right) + \ldots\right\} \int_0^{2\pi} \left\{1 - \xi\frac{d}{dc} - \zeta\frac{d}{dz'} + \frac{\xi^2}{2}\frac{d^2}{dc^2}\right. \\
&\qquad\qquad \left. + \ldots\right\} \frac{c\,d\phi}{\sqrt{\{\varpi'^2 + c^2 - 2c\varpi' \cos(\phi' - \phi) + z'^2\}}} \quad . \quad (37).
\end{aligned}$$

Retaining only those terms of the second order in the small quantities α, β, γ, δ, which do not involve ϕ',

$$\begin{aligned}
\mathrm{V}_0 &= \pi a^2 \left\{1 + \frac{a^2}{8}\left(\frac{d^2}{dc^2} + \frac{d^2}{dz'^2}\right) + \ldots\right\} \int_0^{\pi} \frac{c\,d\phi}{\sqrt{\{\varpi'^2 + c^2 - 2c\varpi' \cos\phi + z'^2\}}} \\
&\quad - \pi a^2 \left\{1 + \frac{a^2}{8}\left(\frac{d^2}{dc^2} + \frac{d^2}{dz'^2}\right) + \ldots\right\} \left\{\Sigma\,(\alpha_n \sin n\phi' + \beta_n \cos n\phi')\frac{d}{dc}\right. \\
&\quad \left. + \Sigma\,(\gamma_n \sin n\phi' + \delta_n \cos n\phi')\frac{d}{dz'}\right\} \int_0^{\pi} \frac{c \cos n\phi\,d\phi}{\sqrt{\{\varpi'^2 + c^2 - 2c\varpi' \cos\phi + z'^2\}}} \\
&\quad + \pi a^2 \left\{1 + \frac{a^2}{8}\left(\frac{d^2}{dc^2} + \frac{d^2}{dz'^2}\right) + \ldots\right\} \left\{\frac{\alpha_n^2 + \beta_n^2}{4}\frac{d^2}{dc^2} + \frac{\alpha_n\gamma_n + \beta_n\delta_n}{2}\frac{d^2}{dc\,dz}\right. \\
&\quad \left. + \frac{\gamma_n^2 + \delta_n^2}{4}\frac{d^2}{dz'^2}\right\} \int_0^{2\pi} \frac{c\,d\phi}{\sqrt{\{\varpi'^2 + c^2 - 2c\varpi' \cos\phi + z'^2\}}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (38).
\end{aligned}$$

Write

$$\left.\begin{aligned} \xi' &= \alpha_n \sin n\phi' + \beta_n \cos n\phi' \\ \zeta' &= \gamma_n \sin n\phi' + \delta_n \cos n\phi' \end{aligned}\right\}$$

Then

$$\begin{aligned} \mathrm{V}_0 = \pi a^2 \bigg\{ 1 + \frac{a^2}{8}\left(\frac{d^2}{dc^2} + \frac{d^2}{dz'^2}\right) \\ + \ldots \bigg\} \left\{ 1 - \xi' \frac{d}{dc} - \zeta' \frac{d}{dz'} + \frac{\xi'^2}{2}\frac{d^2}{dc^2} + \ldots \right\} \int_0^{2\pi} \frac{c \cos\phi \, d\phi}{\sqrt{\{\varpi'^2 + c^2 - 2c\varpi' \cos\phi + z'^2\}}} \\ + \pi a^2 \left\{ 1 + \frac{a^2}{8}\left(\frac{d^2}{dc^2} + \frac{d^2}{dz'^2}\right) + \ldots \right\} \left\{ \xi' \frac{d}{dc} + \zeta' \frac{d}{dz'} \right\} \int_0^{2\pi} \frac{(1 - \cos n\phi)\, d\phi}{\sqrt{\{\varpi'^2 + c^2 - 2c\varpi' \cos\phi + z'^2\}}}. \end{aligned}$$

Let the point ϖ', z', ϕ' be on the surface of the ring.
Let

$$\begin{aligned} \varpi' &= c - \mathrm{R}' \cos\chi' & \text{and} \qquad z' &= \mathrm{R}' \sin\chi' \\ &= c - a\cos\chi_1 - \xi' & &= a \sin\chi_1 + \zeta'. \end{aligned}$$

Retaining only the most important terms,

$$\begin{aligned} \mathrm{V}_0 &= 2\pi a^2 \left(1 - \xi' \frac{d}{dc} - \zeta' \frac{d}{dz'} + \ldots\right) \left\{ \log \frac{8c}{\mathrm{R}'} + \&\text{c.} \right\} \\ &\quad + 2\pi a^2 \left(\xi' \frac{d}{dc} + \zeta' \frac{d}{dz'}\right) \left\{ 2f(n) + f(n) \frac{\mathrm{R}' \cos\chi'}{c} + \ldots \right\} \\ &= 2\pi a^2 \left\{ \log \frac{8(c - \xi')}{a} + \frac{\xi'}{c} f(n) \right\} \\ &= 2\pi a^2 \left\{ \log \frac{8c}{a} + \frac{\xi'}{c} [f(n) - 1] - \frac{\xi'^2}{2c^2} \right\} \\ &= 2\pi a^2 \left\{ \log \frac{8c}{a} + \Sigma [f(n) - 1] (\alpha_n \sin n\phi' + \beta_n \cos n\phi') - \Sigma \frac{\alpha_n^2 + \beta_n^2}{4} \right\}. \quad (39). \end{aligned}$$

§ 22. Inside the ring the potential is V_i, where

$$\mathrm{V}_i = 2\pi a^2 \left\{ \log \frac{8c}{a} + \tfrac{1}{2}\left(1 - \frac{\mathrm{R}^2}{a^2}\right) + \Sigma [f(n) - 1] (\alpha_n \sin n\phi' + \beta_n \cos n\phi') - \Sigma \frac{\alpha_n^2 + \beta_n^2}{4} \right\},$$

where R_1 is the distance from the curve of centres.

The exhaustion of potential energy

$$\mathrm{U} = \tfrac{1}{2} \int \mathrm{V}_i \, dm,$$

thus

$$\frac{\mathrm{U}}{\pi a^2} = \iiint \frac{\mathrm{V}i}{2\pi a^2} (c - x - \xi') \, dx \, dz \, d\phi',$$

or

$$U = 2\pi^3 a^4 c \left\{ \log \frac{8c}{a} + \tfrac{1}{4} - \Sigma \frac{\alpha_n^2 + \beta_n^2}{4} - \Sigma [f(n) - 1] \frac{\alpha_n^2 + \beta_n^2}{2} \right\}$$

$$= \frac{M^2}{2\pi c} \left\{ \log \frac{8c}{a} + \tfrac{1}{4} - \Sigma \frac{\alpha_n^2 + \beta_n^2}{2} [f(n) - \tfrac{1}{2}] \right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (40).$$

For

$$M = \iiint (c - x - \xi')\, dx\, dz\, d\phi'$$
$$= 2\pi^2 a^2 c.$$

§ 23. The exhaustion of potential energy is therefore diminished when the ring is displaced so that the central circle does not remain circular. For this kind of disturbance therefore, the ring is stable, even when the fluid does not rotate. The effect of rotation is to increase this stability.

We have therefore the following results.

The annular form of equilibrium of rotating gravitating fluid is stable for disturbances symmetrical about the axis, and for disturbances which alter the shape of the central curve, but is unstable for long beaded disturbances.

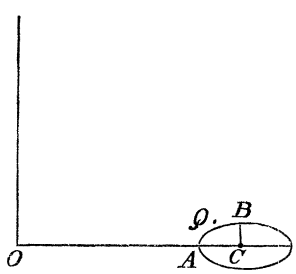
This result was, perhaps, to be expected, as by means of beaded waves, the mass would naturally be broken up into spheroidal masses.

Section III.

§ 24. The methods given in my paper, *ante*, may be used to find the potential of any ring whose cross-section does not deviate far from a circle. They will not, however, apply to a ring whose cross-section is very elliptic, any more than the potential of an elliptic cylinder can be obtained as an approximation from one whose section is circular. In this section, the potential of a ring of elliptic cross-section is obtained by taking the known result for an elliptic cylinder as a first approximation. *The value of the potential obtained applies only to points not far from the surface of the ring.* The potential at internal points may be derived from this, while the potential at other points may be found easily by other methods.

Consider a ring, whose cross-section is elliptic, the major axis of the ellipse being perpendicular to the axis of the ring.

Let the figure represent a section through the axis of the ring.



Let $OC = c : CA = a : CB = b$.

Let x and z be the coordinates of any point in this section of the ring referred to CA and CB as axes.

The potential of the ring at any external point ϖ', z', is

$$\iiint \frac{(c-x)\,dx\,dz\,d\phi}{\sqrt{\{(c-x)^2 + \varpi'^2 - 2\varpi'(c-x)\cos\phi + (z'-z)^2\}}} \quad . \quad . \quad . \quad . \quad . \quad (41),$$

$$= \iint e^{-x(d/dc) - z(d/dz')}\,dx\,dz \int_0^{2\pi} \frac{c\,d\phi}{\sqrt{(\varpi'^2 + c^2 - 2c\varpi'\cos\phi + z'^2)}} \quad . \quad . \quad . \quad (42).$$

Writing D for d/dc, and D′ for d/dz', and taking the double integral over the area of the ellipse, we obtain for the potential at an external point,

$$V_0 = 4\pi ab\left\{\tfrac{1}{2} + \tfrac{1.1}{2.4}\frac{a^2D^2 + b^2D'^2}{2!} + \tfrac{1.1.3}{2.4.6}\frac{(a^2D^2 + b^2D'^2)^2}{4!} + \ldots\right\}\int_0^{\pi}\frac{c\,d\phi}{\sqrt{(\varpi'^2 + c^2 - 2c\varpi'\cos\phi + z'^2)}} \quad . \quad . \quad . \quad (43).$$

§ 25. When the cross-section is circular, the formula may be simplified. Calling

$$\int_0^{\pi}\frac{d\phi}{\sqrt{(\varpi'^2 + c^2 - 2c\varpi'\cos\phi + z'^2)}},\ \mathrm{I}.$$

$$\frac{d^2\mathrm{I}}{dc^2} + \frac{d^2\mathrm{I}}{dz'^2} + \frac{1}{c}\frac{d\mathrm{I}}{dc} = 0.$$

Therefore

$$\left(\frac{d^2}{dc^2} + \frac{d^2}{dz'^2}\right)c\mathrm{I} = 2\frac{d\mathrm{I}}{dc} - \frac{d\mathrm{I}}{dc} = \frac{d\mathrm{I}}{dc}$$

$$\left(\frac{d^2}{dc^2} + \frac{d^2}{dz'^2}\right)^2 c\mathrm{I} = \frac{d}{dc}\left(\frac{d^2}{dc^2} + \frac{d^2}{dz'^2}\right)\mathrm{I} = -\frac{d}{dc}\left(\frac{1}{c}\frac{d\mathrm{I}}{dc}\right)$$

$$\left(\frac{d^2}{dc^2} + \frac{d^2}{dz'^2}\right)^3 c\mathrm{I} = -\frac{d}{dc}\left(\frac{d^2}{dc^2} + \frac{d^2}{dz'^2}\right)\left(\frac{1}{c}\frac{d\mathrm{I}}{dc}\right)$$

$$= -\frac{d}{dc}\left\{\frac{1}{c}(D^2 + D'^2\frac{d\mathrm{I}}{dc} - \frac{2}{c}\frac{d}{dc}\left(\frac{1}{c}\frac{d\mathrm{I}}{dc}\right)\right\}$$

$$= +1.3\frac{d}{dc}\left(\frac{1}{c}\frac{d}{dc}\right)^2\mathrm{I}.$$

So that when the cross-section is circular, we arrive at the formula given in my former paper, p. 59.

$$V_0 = \frac{M}{\pi c}\left\{1 + \frac{a^2}{8c}\frac{d}{dc} - \frac{a^4}{192}\left(\frac{1}{c}\frac{d}{dc}\right)^2 - \ldots\right\}\int_0^{\pi}\frac{c\,d\phi}{\sqrt{(\varpi'^2 + c^2 - 2c\varpi'\cos\phi + z'^2)}} \quad . \quad (44).$$

§ 26. Let the external point at which the potential is required be near the ring; let its coordinates referred to CA and CB be $\mathrm{R}\cos\chi$ and $\mathrm{R}\sin\chi$.

The integral

$$\int_0^\pi \frac{c\,d\phi}{\sqrt{(\varpi'^2 + c^2 - 2c\varpi'\cos\phi + z'^2)}}$$

may be expanded in the form

$$\log\frac{8c}{\mathrm{R}} + \frac{\log\frac{8c}{\mathrm{R}} - 1}{2}\,\frac{\mathrm{R}\cos\chi}{c} + \left(\frac{2\log\frac{8c}{\mathrm{R}} - 3}{16} + \frac{3\log\frac{8c}{\mathrm{R}} - 4}{16}\cos 2\chi\right)\frac{\mathrm{R}^2}{c^2} + \&\text{c.} \quad (45).$$

Now

$$\left\{\tfrac{1}{2} + \tfrac{1.1}{2.4}\,\frac{a^2\mathrm{D}^2 + b^2\mathrm{D}'^2}{2\,!} + \&\text{c.}\right\}\log\frac{8c}{\mathrm{R}}$$

$$= \tfrac{1}{2}\log 8c - \tfrac{1.1}{2.4}\,\tfrac{1}{2}\,\frac{a^2}{c^2}\,\tfrac{1.1.3}{2.4.6}\,\frac{a^4}{4c^4} - \&\text{c.}$$

$$- \tfrac{1}{2}\log\mathrm{R} + \tfrac{1.1}{2.4}\,\frac{a^2 - b^2}{2}\,\frac{\cos^2\chi}{\mathrm{R}^2} + \tfrac{1.1.3}{2.4.6}\,\frac{(a^2 - b^2)^2}{4}\,\frac{\cos 4\chi}{\mathrm{R}^4} + \&\text{c.} \quad . \quad . \quad (46).$$

Let

$$\frac{\mathrm{R}e^{i\chi}}{\sqrt{(a^2 - b^2)}} = y\,;$$

the second series is the real part of

$$- \tfrac{1}{2}\log\sqrt{a^2 - b^2} - \tfrac{1}{2}\log y + \tfrac{1.1}{2.4}\,\frac{1}{2y^2} + \tfrac{1.1.3}{2.4.6}\,\frac{1}{4y^4} + \ldots$$

Calling this series S,

$$\frac{d\mathrm{S}}{dy} = -\left\{\tfrac{1}{2}\,\frac{1}{y} + \tfrac{1.1}{2.4}\,\frac{1}{y^3} + \tfrac{1.1.3}{2.4.6}\,\frac{1}{y^5} + \ldots\right\}$$

$$= \sqrt{y^2 - 1} - y.$$

Therefore

$$\mathrm{S} = \frac{y\sqrt{y^2 - 1}}{2} - \tfrac{1}{2}\log\left(y + \sqrt{y^2 - 1}\right) - \frac{y^2}{2} + \text{const.}$$

Now, when y is very large

$$\mathrm{S} = -\tfrac{1}{2}\log\sqrt{a^2 - b^2} - \tfrac{1}{2}\log y.$$

Therefore

$$\frac{y^2}{2} - \tfrac{1}{4} - \tfrac{1}{2}\log 2y - \frac{y^2}{2} + \text{const.} = -\tfrac{1}{2}\log\sqrt{a^2 - b^2} - \tfrac{1}{2}\log y.$$

Therefore

$$\text{Const.} = \tfrac{1}{4} + \tfrac{1}{2}\log 2 - \tfrac{1}{2}\log\sqrt{a^2 - b^2}.$$

Put

$$d^2 = a^2 - b^2,$$

$$y = \cosh w = \cosh(u + w).$$

Then

$$S = \frac{\cosh w(\sinh w - \cosh w)}{2} - \tfrac{1}{2}\log(\cosh w + \sinh w) + \tfrac{1}{4} - \tfrac{1}{2}\log\frac{d}{2}$$

$$= -\tfrac{1}{2}w - \tfrac{1}{4}e^{-2w} - \tfrac{1}{2}\log\frac{d}{2}.$$

Taking the real part of this, we find for the first term of the potential

$$\tfrac{1}{2}\log\frac{16c}{d} - \frac{u}{2} - \tfrac{1}{4}e^{-2u}\cos 2v - \frac{a^2}{16c^2}. \quad \text{(46}a\text{)}.$$

Calling the operator

$$\tfrac{1}{2} + \tfrac{1.1}{2.4}\frac{a^2D^2 + b^2D'^2}{2!} + \&c., \; f(D, D').$$

Since

$$f(D, D')\,uv = vf(D, D')\,u + D\,v\frac{\partial f}{\partial D}u + D'\,v\frac{\partial f}{\partial D'}u + \&c.$$

Therefore

$$\begin{aligned} f(D, D')\left(\log\frac{8c}{R} - 1\right)\frac{R\cos\chi}{2c} & \\ = \frac{R\cos\chi}{2c}f(D, D')\left(\log\frac{8c}{R} - 1\right) & \\ + \left(\frac{1}{2c} - \frac{R\cos\chi}{2c^2}\right)\frac{\partial f}{\partial D}\left(\log\frac{8c}{R} - 1\right) & \\ + \&c. & \end{aligned}$$

It is easily shown that

$$\frac{\partial f}{\partial D}\left(\log\frac{8c}{R} - 1\right) = \frac{a^2}{12d}(e^{-3u}\cos 3v - 3e^{-u}\cos v) + \frac{a^2}{8c} \quad \text{(47)}.$$

Retaining only the terms of the highest order

$$\begin{aligned} f(D, D')\left\{\left(\log\frac{8c}{R} - 1\right)\frac{R\cos\chi}{2c}\right\} &= \frac{d\cosh u\cos v}{2c}\left\{\tfrac{1}{2}\log\frac{16c}{d} - \frac{u}{2} - \tfrac{1}{2} - \tfrac{1}{4}e^{-2u}\cos 2v\right\} \\ &\quad + \frac{a^2}{24cd}\{e^{-3u}\cos 3v - 3e^{-u}\cos v\} \\ &= \frac{d}{c}\left\{\left(\frac{\log\frac{16c}{d} - u - 1}{4}\cosh u - \tfrac{1}{16}\cosh u\,e^{-2u} - \frac{a^2}{8d^2}e^{-u}\right)\cos v \right. \\ &\quad \left. + \left(-\tfrac{1}{16}\cosh u\,e^{-2u} + \frac{a^2}{24d^2}e^{-3u}\right)\cos 3v\right\} \quad \text{(47}a\text{)}. \end{aligned}$$

Let

$$\log \frac{16c}{d} - u = \mathrm{K}.$$

Then the potential of an elliptic ring at outside points not far from the ring is

$$2\pi ab \left\{ \mathrm{K} - \tfrac{1}{2} e^{-2u} \cos 2v + \frac{d}{c} \left[\left\{ \frac{\mathrm{K} - 1}{2} \cosh u - \tfrac{1}{8} \cosh u e^{-2u} - \tfrac{1}{4} \cosh^2 u_0 \, e^{-u} \right\} \cos v \right. \right.$$

$$\left. \left. + \left\{ -\tfrac{1}{8} \cosh u e^{-2u} + \tfrac{1}{12} \cosh^2 u_0 \, e^{-3u} \right\} \cos 3v \right] \right\} \quad . \quad . \quad . \quad . \quad . \quad (48).$$

At the surface of the ring $u = u_0$ and $\mathrm{K} = \mathrm{K}_0 = \log(16c/d) - u_0$, and

$$\mathrm{V} = 2\pi ab \left\{ \mathrm{K}_0 - \tfrac{1}{2} e^{-2u_0} \cos 2v + \frac{d}{c} \left[\left(\frac{\mathrm{K}_0 - 1}{2} - \frac{1 + 2e^{-2u_0}}{8} \right) \cosh u_0 \cos v \right. \right.$$

$$\left. \left. - \frac{2 - e^{-2u_0}}{24} e^{-2u_0} \cosh u_0 \cos 3v \right] \right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (49).$$

The complexity of the expressions renders it useless to find the potential any further, in the general case: but for a very flat ring, *i.e.*, where b is negligible compared with a, the expressions take much simpler forms.

§ 27. For a very flat ring

$$\mathrm{V} = 4\pi \, ab \left\{ \tfrac{1}{2} + \tfrac{1.1}{2.4} \frac{a^2 \mathrm{D}^2}{2} + \tfrac{1.1.3}{2.4.6} \frac{a^4 \mathrm{D}^4}{4!} + \ldots \right\} \left\{ \log \frac{8c}{\mathrm{R}} + \frac{\log \frac{8c}{\mathrm{R}} - 1}{2} \frac{\mathrm{R} \cos \chi}{c} + \ldots \right\} \quad (50)$$

$$f(\mathrm{D}) \log \frac{8c}{\mathrm{R}} = \tfrac{1}{2} \left(\log \frac{16c}{a} - u \right) - \tfrac{1}{4} e^{-2u} \cos 2v - \frac{a^2}{16c},$$

$$f'(\mathrm{D}) \log \frac{8c}{\mathrm{R}} = \frac{a}{12} \left(e^{-3u} \cos 3v - 3e^{-u} \cos v \right) + \frac{a^2}{8c},$$

$$\tfrac{1}{2} f''(\mathrm{D}) \log \frac{8c}{\mathrm{R}} = \frac{a^2}{16} \left(\log \frac{16c}{a} - u - \tfrac{1}{4} e^{-4u} \cos 4v \right)$$

Therefore,

$$\begin{aligned}\frac{V}{4\pi ab} = \frac{K}{2} &- \tfrac{1}{4}e^{-2u}\cos 2v - \frac{a^2}{16c^2} \\ &+ \left(\frac{K-1}{2} - \tfrac{1}{4}e^{-2u}\cos 2v\right)\frac{a\cosh u\cos v}{2c} \\ &+ \left[\frac{a}{12}(e^{-3u}\cos 3v - 3e^{-u}\cos v) + \frac{a^2}{8c}\right]\left[\frac{1}{2c} - \frac{a\cosh u\cos v}{2c^2}\right] \\ &- \frac{a^2}{32c^2}(K - \tfrac{1}{4}e^{-4u}\cos 4v) \\ &+ \frac{a^2}{c^2}\left(\frac{K}{2} - \tfrac{1}{4}e^{-2u}\cos 2v\right)\frac{5\cosh^2 u\cos^2 v - \sinh^2 u\sin^2 v}{16} \\ &- \frac{a^2}{32c^2}(11\cosh^2 u\cos^2 v - 5\sinh^2 u\sin^2 v) \\ &+ \frac{a^2}{12c^2}(e^{-3u}\cos 3v - 3e^{-u}\cos v)\frac{10\cosh u\cos v}{16} - \frac{11a^2}{128c^2} \\ &+ \frac{a^2}{16c^2}(K - \tfrac{1}{4}e^{-4u}\cos 4v)\tfrac{10}{16}.\end{aligned}$$

Therefore

$$\begin{aligned}V/2\pi ab = K &- \tfrac{1}{2}e^{-2u}\cos 2v \\ &+ \frac{a}{c}\left\{\cos v\left[\frac{K}{2}\cosh u - \frac{4e^{2u}+9+e^{-2u}}{16}e^{-u}\right] - \frac{3e^{u}-e^{-u}}{48}e^{-2u}\cos 3v\right\} \\ &+ \frac{a^2}{c^2}\left\{K\frac{4\cosh 2u+7}{64} - \tfrac{3}{16}e^{2u} - \tfrac{339}{256} - \tfrac{7}{32}e^{-2u} - \tfrac{3}{256}e^{-4u}\right\} \\ &+ \frac{a^2}{c^2}\cos 2v\left\{K\frac{3\cosh 2u+2}{32} + \tfrac{1}{2}e^{2u} + \tfrac{11}{32} + \tfrac{107}{192}e^{-2u} + \tfrac{1}{96}e^{-4u}\right\} \\ &+ \frac{a^2}{c^2}\cos 4v\left\{-\tfrac{3}{256} - \tfrac{1}{96}e^{-2u} - \tfrac{1}{96}e^{-4u}\right\} \quad \ldots \quad (51).\end{aligned}$$

At the surface of the ring

$$u = 0, \quad K = \log\frac{16c}{a},$$

and

$$\begin{aligned}V = 2\pi ab\Big\{K\left(1 + \tfrac{11}{64}\frac{a^2}{c^2}\right) &- \tfrac{323}{128}\frac{a^2}{c^2} + \cos v\left(\frac{K}{2} - \tfrac{7}{8}\right)\frac{a}{c} \\ &+ \cos 2v\left[-\tfrac{1}{2} + \left(\frac{5K}{32} + \tfrac{271}{192}\right)\frac{a^2}{c^2}\right] - \frac{a}{24c}\cos 3v - \tfrac{25}{768}\frac{a^2}{c^2}\cos 4v\Big\} \quad . \quad . \quad (52).\end{aligned}$$

§ 28. Let us take Saturn to be a sphere of radius r and of density ρ. Assume that the ring is fluid rotating with angular velocity ω, of density σ; let its section be supposed elliptic (semi-axes a and b), b being much smaller than a: and let its mean radius be c.

It is necessary that

$$2\pi\sigma ab\left\{\mathrm{K}+\ldots+\frac{a}{c}\left(\frac{\mathrm{K}}{2}-\tfrac{7}{8}\right)\cos v+\left[-\tfrac{1}{2}+\left(\frac{5\mathrm{K}}{12}+\tfrac{271}{192}\right)\frac{a^2}{c^2}\right]\cos 2v\right.$$

$$\left.-\frac{a}{24c}\cos 3v-\tfrac{25}{768}\frac{a^2}{c^2}\cos 4v\right\}$$

$$+\tfrac{4}{3}\pi\rho r^3\left\{\frac{1}{c}+\frac{a\cos v}{c^2}+\frac{a^2\cos^2 v}{c^3}+\ldots\right\}$$

$$+\frac{\omega^2}{2}\{c^2-2ac\cos v+a^2\cos^2 v\}\quad\ldots\ldots\ldots\ldots\quad(53)$$

should be constant at the surface of the ring.

Therefore

$$\left.\begin{aligned}&\pi\sigma\frac{a^2b}{c}\frac{4\mathrm{K}-7}{4}+\tfrac{4}{3}\pi\rho\frac{ar^3}{c^2}-ac\omega^2=0\\ \text{and}\quad&2\pi\sigma ab\left\{-\tfrac{1}{2}+\frac{80\mathrm{K}+271}{192}\frac{a^2}{c^2}\right\}+\tfrac{2}{3}\pi\rho\frac{a^2r^3}{c^3}+\frac{a^2\omega^2}{4}=0\end{aligned}\right\}\quad\ldots\quad(54).$$

Thus

$$\left.\begin{aligned}\frac{\omega^2}{\pi}&=\frac{4\mathrm{K}-7}{4}\frac{b}{a}\frac{a^2}{c^2}\sigma+\tfrac{4}{3}\frac{r^3}{c^3}\rho\\ \text{and also}\quad&=4\quad 1-\frac{80\mathrm{K}+271}{96}\frac{a^2}{c^2}\Big\}\sigma-\tfrac{8}{3}\frac{r^3}{c^3}\rho\end{aligned}\right\}$$

As a rough approximation to Saturn's rings, take

$$\frac{r}{c}=\tfrac{1}{2}\text{ and }\frac{a}{c}=\tfrac{1}{4}.\quad\text{Then }\mathrm{K}=\log_e 64=4{\cdot}15$$

and

$$\left.\begin{aligned}\frac{\omega^2}{\pi}&=\ {\cdot}15\frac{b}{a}\sigma+\tfrac{1}{6}\rho\\ \text{and also}\quad&=2{\cdot}44\frac{b}{a}\sigma-\tfrac{1}{3}\rho\end{aligned}\right\}\quad\ldots\ldots\ldots\ldots\quad(55),$$

therefore

$$\rho/\sigma=4{\cdot}58\,b/a.$$

For a ring, about the thickness of Saturn's, we shall have roughly $\rho/\sigma=\frac{1}{100}$; or the ring would need to be 100 times as dense as the planet, if it were a continuous fluid mass.

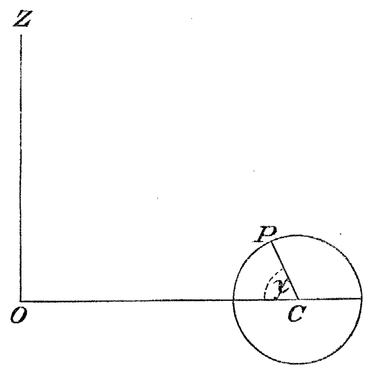
SECTION IV.

§ 29. The motion of two or more vortex rings on the same axis is most easily discussed by means of the stream-line function. The determination of this function is easily reduced to the determination of the potential of a distribution of gravitating matter, so that the preceding methods apply to this problem.

The Steady Motion of a Vortex ring of Finite Cross-section.

As before, let the figure represent a section through the axis of the ring.

Let O be the centre of the ring, Oz the axis, C the centre of gravity of the cross-section.



Let $OC = c$: the cross-section is nearly circular; let its equation be

$$R = a\,(1 + \beta_1 \cos\chi + \beta_2 \cos 2\chi + \&c.) \quad . \quad . \quad . \quad . \quad . \quad (56),$$

where β_1, β_2, &c., are small quantities.

Since C is the centre of gravity of the cross-section

$$\int_0^{2\pi}\int_0^{R} R\cos\chi\, R\, dR\, d\chi = 0.$$

Therefore

$$\int_0^{2\pi} (1 + \beta_1 \cos\chi + \beta_2 \cos 2\chi + \ldots)^3 \cos\chi\, d\chi = 0,$$

or

$$\beta_1 + \beta_1\,\beta_2 + \beta_2\,\beta_3 + \beta_3\,\beta_4 + \ldots = 0 \quad . \quad . \quad . \quad . \quad . \quad (57).$$

Now it will be shown that

β_2 is of the second order in a/c;
β_3 of the third, &c.

Thus β_1 will be of the fifth order.

Therefore, *correctly to the fourth order*, the equation of the cross-section is

$$R = a\{1 + \beta_2 \cos 2\chi + \beta_3 \cos 3\chi + \beta_4 \cos 4\chi\} \quad . \quad . \quad . \quad . \quad (58).$$

§ 30. Let ω be the molecular rotation, and Ψ the stream-line function.

Then

$$\frac{d^2\Psi}{dz^2} + \frac{d^2\Psi}{d\varpi^2} - \frac{1}{\varpi}\frac{d\Psi}{d\varpi} = 0$$

outside the ring, and

$$\frac{d^2\Psi}{dz^2} + \frac{d^2\Psi}{d\varpi^2} - \frac{1}{\varpi}\frac{d\Psi}{d\varpi} + 2\varpi\omega = 0$$

inside the ring, while Ψ and $d\Psi/dn$ are continuous at the surface.

Let the *vorticity* be constant throughout the ring.

Then $\omega = (M/2)\,\varpi$, where M is a constant.

Write $\Psi = \chi\varpi$, and the above equations become

$$\left.\begin{aligned} &\frac{d^2\chi}{dz^2} + \frac{d^2\chi}{d\varpi^2} + \frac{1}{\varpi}\frac{d\chi}{d\varpi} - \frac{\chi}{\varpi^2} = 0 \\ \text{and} \quad &\frac{d^2\chi}{dz^2} + \frac{d^2\chi}{d\varpi^2} + \frac{1}{\varpi}\frac{d\chi}{d\varpi} - \frac{\chi}{\varpi^2} + M\varpi = 0 \end{aligned}\right\} .$$

Therefore $\chi \cos \phi$ is the potential of matter of density $(M\varpi \cos \phi)/4\pi$ occupying the same space as the ring.

Therefore, at any external point, ϖ', z', ϕ',

$$\chi \cos \phi' = \frac{M}{4\pi} \cos \phi' \iiint \frac{\varpi^2 \cos \phi \, d\varpi \, dz \, d\phi}{\sqrt{\{\varpi'^2 - 2\varpi\varpi' \cos \phi + \varpi^2 + (z' - z)^2\}}} .$$

Therefore

$$\Psi = \frac{M}{4\pi}\varpi' \iiint \frac{\varpi^2 \cos \phi \, d\varpi \, dz \, d\phi}{\sqrt{\{\varpi'^2 - 2\varpi\varpi' \cos \phi + \varpi^2 + (z' - z)^2\}}} \quad . \quad . \quad . \quad . \quad (59),$$

the integration extending all over the ring.

(The value of Ψ at an internal point may be found as the solution of

$$\frac{d^2\Psi}{dz^2} + \frac{d^2\Psi}{d\varpi^2} - \frac{1}{\varpi}\frac{d\Psi}{d\varpi} + M\varpi^2 = 0,$$

which gives values continuous at the surface).

Let $\varpi = c - x$.

Then

$$\Psi = \frac{\mathrm{M}}{4\pi}\varpi' \iiint \frac{(c-x)^2 \cos\phi \, dx \, dz \, d\phi}{\sqrt{\{\varpi'^2 - 2\varpi'(c-x)\cos\phi + (c-x)^2 + (z'-z)^2\}}}$$

$$= \frac{\mathrm{M}}{4\pi}\varpi' \iint e^{-x(d/dc) - z(d/dz')} \, dx \, dz \int_0^{2\pi} \frac{c^2 \cos\phi \, d\phi}{\sqrt{\{\varpi'^2 - 2c\varpi'\cos\phi + c^2 + z'^2\}}} \quad . \quad . \quad . \quad (60),$$

where the double integral is to be taken all over the area of the cross-section $\mathrm{R} = a(1 + \beta_2 \cos 2\chi + \beta_3 \cos 3\chi + \beta_4 \cos 4\chi)$.

§ 31. While finding this double integral we may treat d/dc and d/dz' as constants. Put

$$\frac{d}{dc} = \nabla \cos\alpha \; \frac{d}{dz'} = \nabla \sin\alpha.$$

Then

$$\nabla^2 = \frac{d^2}{dc^2} + \frac{d^2}{dz'^2}.$$

The double integral

$$\iint e^{-\nabla(x\cos\alpha + z\sin\alpha)} \, dx \, dz$$

taken over the area of

$$\mathrm{R} = a(1 + \beta_2 \cos 2\chi + \beta_3 \cos 3\chi + \beta_4 \cos 4\chi)$$

may be conveniently calculated in three parts.

(1) Over the circle of radius a;

(2) Round the circumference of this circle to obtain the first powers of β_2, β_3, β_4;

(3) To find separately the coefficient of β_2^2.

Part (1)

$$= \iint e^{-\nabla(x\cos\alpha + z\sin\alpha)} \, dx \, dz,$$

where

$$x^2 + z^2 < a^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (61),$$

$$= a^2 \iint e^{-\nabla\zeta} \, d\xi \, d\zeta,$$

where

$$\xi^2 + \zeta^2 < 1,$$

$$= \pi a^2 \int_{-1}^{1} e^{-\nabla\zeta} \sqrt{1 - \zeta^2} \, d\zeta$$

$$= 4a^2 \int_0^{\frac{\pi}{2}} e^{-a\nabla \sin\theta} \cos^2\theta \, d\theta$$

$$= \pi a^2 \left\{1 + \frac{a^2\nabla^2}{2^2 . 2} + \frac{a^4\nabla^4}{2^2 . 4^2 . 3} + \ldots\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (62).$$

Part (2)

$$= a^2 \int_0^{2\pi} e^{-a\nabla \cos(\chi - a)} (\beta_2 \cos 2\chi + \beta_3 \cos 3\chi + \beta_4 \cos 4\chi)\, d\chi$$

$$= a^2 \int_0^{2\pi} \{\beta_2 \cos 2\alpha \cos 2\psi + \beta_3 \cos 3\alpha \cos 3\psi + \beta_4 \cos 4\alpha \cos 4\psi\}\, e^{-a\nabla \cos\psi}\, d\psi$$

$$= \pi a^4 \frac{\beta_2}{4}\left(\frac{d^2}{dc^2} - \frac{d^2}{dz'^2}\right)\left\{1 + \frac{a^2\nabla^2}{12} + \frac{a^4\nabla^4}{384} + \dots\right\}$$
$$- \pi a^5 \frac{\beta_3}{12}\left(\frac{d^3}{dc^3} - 3\frac{d}{dc}\frac{d^2}{dz'^2}\right)\left(1 + \frac{a^2\nabla^2}{16}\right)$$
$$+ \pi a^6 \frac{\beta_6}{192}\left(\frac{d^4}{dc^4} - 6\frac{d^2}{dc^2}\frac{d^2}{dz'^2} + \frac{d^4}{dz'^4}\right) \quad \dots \quad (63).$$

Part (3) is easily found to be

$$\pi a^2 \frac{\beta_2^2}{2} + \tfrac{5}{16}\pi a^2 \beta_2^2 \frac{a^4}{4!}\left(\frac{d^4}{dc^4} - 6\frac{d^2}{dc^2}\frac{d^2}{dz'^2} + \frac{d^4}{dz'^4}\right) \quad \dots \quad (64).$$

At any external point the stream-function is, therefore, the effect of the operation

$$\pi a^2 \left\{1 + \frac{\beta_2^2}{2} + \frac{a^2\nabla^2}{2^2.2} + \frac{a^4\nabla^4}{2^2.4^2.3} + \dots\right\} + \pi a^4 \frac{\beta_2}{4}\left\{\frac{d^2}{dc^2} - \frac{d^2}{dz'^2}\right\}\left\{1 + \frac{a^2\nabla^2}{12} + \frac{a^4\nabla^4}{384} + \dots\right\}$$
$$- \pi a^5 \frac{\beta_3}{12}\left(\frac{d^3}{dc^3} - 3\frac{d}{dc}\frac{d^2}{dz'^2}\right)\left(1 + \frac{a^2\nabla^2}{16} + \dots\right)$$
$$+ \pi a^6 \left(\frac{\beta_4}{192} + \frac{5\beta_2^2}{16\lfloor 4}\right)\left(\frac{d^4}{dc^4} - 6\frac{d^2}{dc^2}\frac{d^2}{dz'^2} + \frac{d^4}{dz'^4}\right) \quad \dots \quad (65)$$

on

$$\int_0^{2\pi} \frac{c^2 \cos\phi\, d\phi}{\sqrt{(\varpi'^2 - 2\varpi' c\cos\phi + c^2 + z'^2)}}.$$

§ 32. Now at the surface of the ring

$$\int_0^{2\pi} \frac{c^2 \cos\phi\, d\phi}{\sqrt{(\varpi'^2 - 2\varpi' c\cos\phi + c^2 + z'^2)}} \text{ is of the order } \frac{c^2}{a},$$

$$a^n \frac{d^n}{dc^n}\int_0^{2\pi} \frac{c^2 \cos\phi\, d\phi}{\sqrt{(\varpi'^2 - 2\varpi' c\cos\phi + c^2 + z'^2)}} \text{ is of the order } a^n \frac{c^2}{a^{n+1}} \text{ i.e., } \frac{c^2}{a},$$

$$a^{2n}\left(\frac{d^2}{dc^2} + \frac{d^2}{dz'^2}\right)^n \int_0^{2\pi} \frac{c^2 \cos\phi\, d\phi}{\sqrt{(\varpi'^2 - 2\varpi' c\cos\phi + c^2 + z'^2)}}$$

will be shown to be of the order

$$a^{2n} \frac{c^2}{ac^{n+1}\, c^n} \quad i.e., \quad \frac{c^2}{a} \left(\frac{a}{c}\right)^n .$$

We have assumed β_2 to be of the second order, β_3 of the third, &c. Therefore, in the above formula, all terms as far as $\left(\frac{a}{c}\right)^4$ have been retained.

§ 33. Let

$$\mathrm{I} = \int_0^\pi \frac{c \cos \phi \, d\phi}{\sqrt{(\varpi'^2 - 2\varpi' c \cos \phi + c^2 + z'^2)}} .$$

Then

$$\frac{d^2 \mathrm{I}}{dz'^2} + \frac{d^2 \mathrm{I}}{dc^2} - \frac{1}{c} \frac{d\mathrm{I}}{dc} = 0 .$$

Let

$$\mathrm{U} = \mathrm{I} \, . \, c.$$

Then

$$\left(\frac{d^2}{dc^2} + \frac{d^2}{dz'^2}\right) \mathrm{U} = c \left(\frac{d^2 \mathrm{I}}{dc^2} + \frac{2}{c} \frac{d\mathrm{I}}{dc} + \frac{d^2 \mathrm{I}}{dz'^2}\right) .$$

Or,

$$\nabla^2 \mathrm{U} = 3 \frac{d\mathrm{I}}{dc} ,$$

$$\nabla^4 \mathrm{U} = 3 \frac{d}{dc} \nabla^2 \mathrm{I}$$

$$= 3 \, . \, 1 \frac{d}{dc} \left(\frac{1}{c} \frac{d\mathrm{I}}{dc}\right) ,$$

$$\nabla^6 \mathrm{U} = 3 \, . \, 1 \frac{d}{dc} \nabla^2 \left(\frac{1}{c} \frac{d\mathrm{I}}{dc}\right)$$

$$= 3 \, . \, 1 \frac{d}{dc} \left\{\frac{1}{c} \nabla^2 \frac{d\mathrm{I}}{dc} - \frac{2}{c} \frac{d}{dc} \left(\frac{1}{c} \frac{d\mathrm{I}}{dc}\right)\right\} ,$$

$$= 3 \, . \, 1 \, (-1) \frac{d}{dc} \left(\frac{1}{c} \frac{d}{dc}\right)^2 \mathrm{I} .$$

Hence

$$\nabla^{2n} \mathrm{U} = 3 \, . \, 1 \, (-1) \ldots (-2n + 5) \frac{d}{dc} \left(\frac{1}{c} \frac{d}{dc}\right)^{n-1} \mathrm{I} \quad . \quad . \quad . \quad (65\text{A}).$$

§ 34. Therefore at a point ϖ', z', ϕ', the stream-function is given by

$$\Psi = \frac{\mathrm{M}a^2c}{2}\varpi'\left\{1 + \frac{\beta_2^2}{2} + \frac{3a^2}{8}\frac{1}{c}\frac{d}{dc} + \frac{a^4}{64}\left(\frac{1}{c}\frac{d}{dc}\right)^2 - \frac{a^6}{3072}\left(\frac{1}{c}\frac{d}{dc}\right)^3 + \frac{a^8}{81920}\left(\frac{1}{c}\frac{d}{dc}\right)^4\right\}\mathrm{I}$$
$$+ \frac{\mathrm{M}a^2}{2}\varpi'\frac{a^2\beta_2}{4}\left(\frac{d^2}{dc^2} - \frac{d^2}{dz'^2}\right)c\left\{\mathrm{I} + \frac{a^2}{4}\frac{1}{c}\frac{d\mathrm{I}}{dc} + \frac{a^4}{128}\left(\frac{1}{c}\frac{d}{dc}\right)^2\right\}\mathrm{I}$$
$$- \frac{\mathrm{M}a^2}{2}\varpi'\frac{a^3\beta_3}{24}\left(\frac{d^3}{dc^3} - 3\frac{d}{dc}\frac{d^2}{dz'^2}\right)c\left(\mathrm{I} + \frac{3a^2}{16}\frac{1}{c}\frac{d\mathrm{I}}{dc}\right)$$
$$+ \frac{\mathrm{M}a^2}{2}\varpi'\frac{a^4}{4!}\left(\frac{\beta_4}{8} + \frac{5\beta_2^2}{16}\right)\left(\frac{d^4}{dc^4} - 6\frac{d^2}{dc^2}\frac{d^2}{dz'^2} + \frac{d^4}{dz'^4}\right)c\mathrm{I} \quad . \quad . \quad . \quad . \quad (66).$$

Using the fact that $\frac{d^2\mathrm{I}}{dc^2} - \frac{1}{c}\frac{d\mathrm{I}}{dc} + \frac{d^2\mathrm{I}}{dz'^2} = 0$, and neglecting terms of order higher than $\left(\frac{a}{c}\right)^4$ since $\left(\frac{1}{c}\frac{d}{dc}\right)^n\mathrm{I}$ is of order $\frac{1}{a^nc^n}$ at the surface of the ring, we find that

$$\frac{2\Psi}{\mathrm{M}a^2c\varpi'} = \left\{1 + \frac{3a^2}{8}\frac{1}{c}\frac{d}{dc} + \frac{a^4}{64}\left(\frac{1}{c}\frac{d}{dc}\right)^2 - \frac{a^6}{3072}\left(\frac{1}{c}\frac{d}{dc}\right)^3 + \frac{a^8}{81920}\left(\frac{1}{c}\frac{d}{dc}\right)^4\right\}\mathrm{I}$$
$$+ \beta_2\left\{\frac{3a^2}{4}\frac{1}{c}\frac{d}{dc} + \frac{(8c^2 + 5a^2)a^2}{16}\left(\frac{1}{c}\frac{d}{dc}\right)^2 + \frac{a^4c^2}{8}\left(\frac{1}{c}\frac{d}{dc}\right)^3 + \frac{a^6c^2}{256}\left(\frac{1}{c}\frac{d}{dc}\right)^4\right\}\mathrm{I}$$
$$- \beta_3\left\{\frac{5a^3c}{8}\left(\frac{1}{c}\frac{d}{dc}\right)^2 + \frac{a^3c^3}{6}\left(\frac{1}{c}\frac{d}{dc}\right)^3 + \frac{a^5c^3}{32}\left(\frac{1}{c}\frac{d}{dc}\right)^4\right\}\mathrm{I}$$
$$+ \left(\frac{\beta_4}{24} + \frac{5\beta_2^2}{48}\right)a^4c^4\left(\frac{1}{c}\frac{d}{dc}\right)^4\mathrm{I} \quad . \quad . \quad . \quad . \quad (67).$$

Therefore

$$\Psi = \frac{\mathrm{M}a^2c}{2}\varpi'\left\{\alpha_0\mathrm{I} + \alpha_1 ac\frac{1}{c}\frac{d\mathrm{I}}{dc} + \alpha_2 a^2c^2\left(\frac{1}{c}\frac{d}{dc}\right)^2\mathrm{I} + \ldots\right\} \quad . \quad . \quad . \quad (68),$$

where

$$\left.\begin{aligned} \alpha_0 &= 1 + \frac{\beta_2^2}{2} \\ \alpha_1 &= \frac{3\sigma}{8} + \frac{3\sigma}{4}\beta_2 \\ \alpha_2 &= \frac{\sigma^2}{64} + \beta_2\left(\tfrac{1}{2} + \frac{5\sigma^2}{16}\right) - \beta_3\frac{5\sigma}{8} \\ \alpha_3 &= -\frac{\sigma^3}{3.2^{10}} + \beta_2\frac{\sigma}{8} - \frac{\beta_3}{6} \\ \alpha_4 &= \frac{\sigma^4}{5.2^{14}} + \beta_2\frac{\sigma^2}{256} - \beta_3\frac{\sigma}{32} + \frac{\beta_4}{24} + \frac{5\beta_2^2}{48} \\ \text{and} \quad \sigma &= \frac{a}{c} \end{aligned}\right\} \quad . \quad . \quad . \quad (69).$$

§ 35. Let the point ϖ', z', ϕ' be on the surface of the ring, so that

$$\varpi' = c - \mathrm{R}\cos\chi, \qquad z' = \mathrm{R}\sin\chi.$$

Writing l for $\log(8c/\mathrm{R}) - 2$, and s for R/c, we have

$$\left.\begin{aligned}
\mathrm{J} = l + \frac{l-1}{2} s \cos\chi + \left(\frac{6l+1}{16} + \frac{3l-4}{16}\cos 2\chi\right) s^2 \\
+ \left(\frac{33l+1}{64}\cos\chi + \frac{15l-23}{192}\cos 3\chi\right) s^3 \\
+ \left(\frac{540l+27}{2048} + \frac{240l-23}{768}\cos 2\chi + \frac{105l-176}{3072}\cos 4\chi\right) s^4 + \dots \\
(\textit{Suprà}, \text{ p. } 53).
\end{aligned}\right\}$$

We obtain, by differentiation,

$$\left.\begin{aligned}
\frac{1}{c}\frac{d\mathrm{J}}{dc} = -\frac{1}{c^2 s}\Big\{\cos\chi - \left(\frac{2l+1}{4} - \frac{\cos 2\chi}{4}\right) s - \left(\frac{20l+5}{32}\cos\chi - \tfrac{3}{32}\cos 3\chi\right) s^2 \\
- \left(\frac{36l+3}{128} + \frac{24l+3}{64}\cos 2\chi - \tfrac{5}{128}\cos 4\chi\right) s^3 + \dots\Big\} \\
\left(\frac{1}{c}\frac{d}{dc}\right)^2 \mathrm{J} = \frac{1}{c^4 s^2}\Big\{\cos 2\chi + \left(\frac{\cos\chi}{4} + \frac{\cos 3\chi}{4}\right) s + \left(-\frac{12l+5}{32} + \tfrac{3}{32}\cos 4\chi\right) s^2 + \dots\Big\} \\
\left(\frac{1}{c}\frac{d}{dc}\right)^3 \mathrm{J} = -\frac{1}{c^6 s^3}\{2\cos 3\chi + (2\cos 2\chi + \tfrac{1}{2}\cos 4\chi) s + \dots\} \\
\left(\frac{1}{c}\frac{d}{dc}\right)^4 \mathrm{J} = \frac{1}{c^8 s^4}\{6\cos 4\chi + \dots\}
\end{aligned}\right\} (70).$$

Multiplying by ϖ' or $c(1 - s\cos\chi)$

$$\left.\begin{aligned}
\mathrm{J}\frac{\varpi'}{c} = l - \frac{l+1}{2} s \cos\chi + \left(\frac{2l+5}{16} - \frac{l}{16}\cos 2\chi\right) s^2 \\
+ \left(\frac{3l+5}{64}\cos\chi - \frac{3l-1}{192}\cos 3\chi\right) s^3 \\
+ \left(\frac{12l+11}{2048} + \frac{12l+17}{768}\cos 2\chi - \frac{15l-8}{3072}\cos 4\chi\right) s^4 + \dots \\
\frac{1}{c}\frac{d\mathrm{J}}{dc}\frac{\varpi'}{c} = +\frac{1}{c^2 s}\Big\{-\cos\chi + \left(\frac{2l+3}{4} + \frac{\cos 2\chi}{4}\right) s \\
+ \left(\frac{4l+1}{32}\cos\chi + \tfrac{1}{32}\cos 3\chi\right) s^2 \\
+ \left(-\frac{4l+7}{128} + \frac{4l+1}{64}\cos 2\chi + \tfrac{1}{128}\cos 4\chi\right) s^3 + \dots\Big\} \\
\left(\frac{1}{c}\frac{d}{dc}\right)^2 \mathrm{J}\frac{\varpi'}{c} = \frac{1}{c^4 s^2}\Big\{\cos 2\chi - \left(\frac{\cos\chi}{4} + \frac{\cos 3\chi}{4}\right) s \\
- \left(\frac{12l+9}{32} + \tfrac{1}{4}\cos 2\chi + \tfrac{1}{32}\cos 4\chi\right) s^2 \dots\Big\} \\
\left(\frac{1}{c}\frac{d}{dc}\right)^3 \mathrm{J}\frac{\varpi'}{c} = -\frac{1}{c^6 s^3}\{2\cos 3\chi + (\cos 2\chi - \tfrac{1}{2}\cos 4\chi) s + \dots\} \\
\left(\frac{1}{c}\frac{d}{dc}\right)^4 \mathrm{J}\frac{\varpi'}{c} = +\frac{1}{c^8 s^4}\{6\cos 4\chi + \dots\}.
\end{aligned}\right\} . (71)$$

At the surface of the ring

$$R = a(1 + \beta_2 \cos 2\chi + \&c.).$$

Substituting for s, $\sigma(1 + \beta_2 \cos 2\chi + \ldots)$ and writing $\log \frac{8c}{a} - 2 = \lambda$,

$$\left.\begin{aligned}
J\frac{\varpi'}{c} &= \lambda + \frac{2\lambda + 5}{16}\sigma^2 + \frac{12\lambda + 11}{2048}\sigma^4 - \frac{2\lambda - 1}{32}\sigma^2\beta_2 + \frac{\beta_2^2}{4} \\
&\quad + \sigma\cos\chi\left(-\frac{\lambda + 1}{2} + \frac{3\lambda + 5}{64}\sigma^2 - \frac{\lambda}{4}\beta_2\right) \\
&\quad + \cos 2\chi\left(-\frac{\lambda\sigma^2}{16} + \frac{12\lambda + 7}{768}\sigma^4 - \beta_2 - \frac{\lambda}{4}\sigma\beta_3 + \frac{\lambda + 2}{4}\sigma^2\beta_2\right) \\
&\quad + \cos 3\chi\left(-\frac{3\lambda - 1}{192}\sigma^3 - \beta_3 - \frac{\lambda\sigma\beta_2}{4}\right) \\
&\quad + \cos 4\chi\left(-\frac{15\lambda - 8}{3072}\sigma^4 - \beta_4 - \frac{\lambda}{4}\sigma\beta_3 - \frac{2\lambda - 1}{32}\sigma^2\beta_2 + \frac{\beta_2^2}{4}\right) + \ldots \\
\frac{1}{c}\frac{dJ}{dc}\frac{\varpi'}{c} &= \frac{1}{c^2\sigma}\left\{\frac{2\lambda + 3}{4}\sigma - \frac{4\lambda + 7}{128}\sigma^3 + \cos\chi\left(-1 + \frac{4\lambda + 1}{32}\sigma^2 + \tfrac{1}{2}\beta_2\right)\right. \\
&\quad + \cos 2\chi\left(\frac{\sigma}{4} + \frac{4\lambda + 1}{64}\sigma^3 - \tfrac{1}{2}\sigma\beta_2 + \tfrac{1}{2}\beta_3\right) \\
&\quad \left. + \cos 3\chi\left(\frac{\sigma^2}{32} + \tfrac{1}{2}\beta_2\right) + \cos 4\chi\left(\frac{\sigma^3}{128} + \frac{\beta_3}{2}\right)\right\} \\
\left(\frac{1}{c}\frac{d}{dc}\right)^2 J\frac{\varpi'}{c} &= \frac{1}{c^4\sigma^2}\left\{-\frac{12\lambda + 9}{32}\sigma^2 - \beta_2 - \frac{\sigma}{4}\cos\chi + \cos 2\chi\left(1 - \frac{\sigma^2}{4}\right)\right. \\
&\quad \left. - \frac{\sigma}{4}\cos 3\chi - \cos 4\chi\left(\frac{\sigma^2}{32} + \beta_2\right)\right\} \\
\left(\frac{1}{c}\frac{d}{dc}\right)^3 J\frac{\varpi'}{c} &= \frac{1}{c^6\sigma^3}\left\{-\sigma\cos 2\chi - 2\cos 3\chi - \tfrac{1}{2}\sigma\cos 4\chi\right\} \\
\left(\frac{1}{c}\frac{d}{dc}\right)^4 J\frac{\varpi'}{c} &= \frac{1}{c^8\sigma^4}6\cos 4\chi
\end{aligned}\right\} \quad (72).$$

§ 36. Now, in the steady motion, if V be the velocity of the ring,

$$\Psi - \tfrac{1}{2}V\varpi^2 = \text{constant at the surface of the ring} \quad \ldots \ldots \quad (73).$$

Therefore

$$\Psi - \tfrac{1}{2}V\left(c^2 - 2cR\cos\chi + \frac{R^2}{2} + \frac{R^2}{2}\cos 2\chi\right)$$

is constant when

$$R = a(1 + \beta_2 \cos 2\chi + \ldots).$$

Therefore

$$\Psi - \frac{Vc^2}{2}\left\{1 + \frac{\sigma^2}{2} + \frac{\sigma^2}{2}\beta_2 + \cos\chi(-2\sigma - \sigma\beta_2) + \cos 2\chi\left(\frac{\sigma^2}{2} + \sigma^2\beta_2 - \sigma\beta_3\right)\right.$$
$$\left. - \sigma\beta_2\cos 3\chi + \cos 4\chi\left(-\sigma\beta_3 + \sigma^2\frac{\beta_2}{2}\right)\right\} \quad \ldots \quad (74).$$

is constant.

Equating the coefficients of $\cos\chi$, $\cos 2\chi$, $\cos 3\chi$, $\cos 4\chi$ to zero,

$$\left.\begin{aligned}
&a_0\sigma\left(-\frac{\lambda+1}{2} + \frac{3\lambda+5}{64}\sigma^2 - \frac{\lambda}{4}\beta_2\right) + a_1\left(-1 + \frac{4\lambda+1}{32}\sigma^2 + \frac{\beta_2}{2}\right) \\
&\qquad - a_2\frac{\sigma}{4} + \frac{V}{Ma^2}(2\sigma + \sigma\beta_2) = 0 \\
&a_0\left(-\frac{\lambda\sigma^2}{16} + \frac{12\lambda+7}{768}\sigma^4 - \beta_2 - \frac{\lambda}{4}\sigma\beta_3 + \frac{\lambda+2}{4}\sigma^2\beta_2\right) \\
&\qquad + a_1\left(\frac{\sigma}{4} + \frac{4\lambda+1}{64}\sigma^3 - \frac{\sigma}{2}\beta_2 + \tfrac{1}{2}\beta_3\right) + a_2\left(1 - \frac{\sigma^2}{4}\right) \\
&\qquad - a_3\sigma - \frac{V}{Ma^2}\left(\frac{\sigma^2}{2} + \sigma^2\beta_2 - \sigma\beta_3\right) = 0 \\
&a_0\left(-\frac{3\lambda-1}{192}\sigma^3 - \beta_3 - \frac{\lambda\sigma}{4}\beta_2\right) + a_1\left(\frac{\sigma^2}{32} + \frac{\beta_2}{2}\right) \\
&\qquad - a_2\frac{\sigma}{4} - 2a_3 + \frac{V}{Ma^2}\sigma\beta_2 = 0 \\
&a_0\left(-\frac{15\lambda-8}{3072}\sigma^4 - \beta_4 - \frac{\lambda}{4}\sigma\beta_3 - \frac{2\lambda-1}{32}\sigma^2\beta_2 + \tfrac{1}{4}\beta_2^2\right) \\
&\qquad + a_1\left(\frac{\sigma^3}{128} + \frac{\beta_3}{2}\right) + a_2\left(-\frac{\sigma^2}{32} - \beta_2\right) \\
&\qquad + a_3\frac{\sigma}{2} + 6a_4 + \frac{V}{Ma^2}\left(\sigma\beta_3 - \sigma^2\frac{\beta_2}{2}\right) = 0
\end{aligned}\right\} \quad . \quad (75).$$

Now, $\omega = \frac{1}{2}M\varpi$; therefore, if m be the strength of the vortex,

$$m = \iint \omega\, d\sigma = \tfrac{1}{2}M\iint (c - R\cos\chi)\, R\, dR\, d\chi$$
$$= \tfrac{1}{2}M\pi a^2 c\left(1 + \frac{\beta_2^2}{2}\right) \quad \ldots \quad (76).$$

Substituting then for $M a^2$, and solving the above equations:

$$\left.\begin{aligned}
V &= \frac{m}{\pi c}\left\{\frac{4\lambda+7}{8} - \frac{12\lambda+9}{64}\sigma^2\right\} \\
\beta_2 &= -\frac{12\lambda+7}{32}\sigma^2 + \frac{72\lambda+77}{3 . 2^{10}}\sigma^4 \\
\beta_3 &= -\frac{168\lambda+63}{1024}\sigma^3 \\
\beta_4 &= \frac{84\lambda^2+111\lambda+41}{512}\sigma^4
\end{aligned}\right\} \quad \ldots \quad (77).$$

These equations give the velocity of the ring in steady motion, and the form of its cross-section. The cross-section is slightly elongated in the direction of the motion of the ring. The quantities β_3 and β_4 are very small; for example, when $\sigma = \cdot 3$, the case of a very thick ring,

$$V = 2\cdot 96 \times \frac{m}{2\pi c}, \quad \beta_2 = -\cdot 063, \quad \beta_3 = -\cdot 006, \quad \beta_4 = \cdot 005.$$

§ 37. *Fluted oscillations of a Circular Vortex-Ring.*

Let the cross-section of the ring when disturbed be given by

$$R = a\left\{1 + \Sigma\left(\alpha_n \sin n\chi + \beta_n \cos n\chi\right)\right\}. \quad . \quad . \quad . \quad . \quad . \quad . \quad (78)$$

At an external point, not far from the ring, the stream-line function is given by

$$\Psi = \frac{mc}{\pi}\left\{\log\frac{8c}{R} - 2 - \frac{\log\frac{8c}{R} - 1}{2}\frac{R\cos\chi}{c} - \frac{3a^2}{8}\frac{\cos\chi}{cR}\right\}$$
$$+ \frac{mc}{\pi}\Sigma\frac{a^n}{nR^n}\left(\alpha_n \sin n\chi + \beta_n \cos n\chi\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (79).$$

Let the central circle of the ring have moved a distance z_0 from the plane of x, y. Then

$$\dot{R} + \frac{\partial R}{\partial z_0}\dot{z}_0 + \frac{\partial R}{\partial c}\dot{c} = \dot{a} + \Sigma\left[(\dot{a}\alpha_n + a\dot{\alpha}_n)\sin n\chi + (\dot{a}\beta_n + a\dot{\beta}_n)\cos n\chi\right]$$
$$+ a\,\Sigma n\left(\alpha_n \cos n\chi - \beta_n \sin n\chi\right)\left(\dot{\chi} + \frac{\partial\chi}{\partial z_0}\dot{z}_0 + \frac{\partial\chi}{\partial c}\dot{c}\right) \quad . \quad . \quad (80).$$

Now

$$\frac{\partial R}{\partial c} = \cos\chi, \qquad \frac{\partial\chi}{\partial c} = -\frac{\sin\chi}{R},$$

$$\frac{\partial R}{dz_0} = -\sin\chi, \qquad \frac{\partial\chi}{\partial z_0} = -\frac{\cos\chi}{R},$$

and

$$\dot{R} = \frac{1}{\varpi}\frac{d\Psi}{Rd\chi}, \qquad \dot{\chi} = -\frac{1}{\varpi}\frac{d\Psi}{RdR}.$$

Therefore

$$\dot{R} = \frac{m}{\pi(c - R\cos\chi)}\left\{\frac{\log\frac{8c}{R} - 1}{2}\frac{1}{c}\sin\chi + \frac{3a^2}{8}\frac{1}{cR^2}\sin\chi\right\}$$
$$+ \frac{m}{\pi(c - R\cos\chi)}\frac{1}{R}\Sigma\frac{a^n}{R^2}\left(\alpha_n \cos n\chi - \beta_n \sin n\chi\right) \quad . \quad . \quad . \quad . \quad (81),$$

and

$$\dot{\chi} = \frac{m}{\pi(c - \mathrm{R}\cos\chi)} \frac{1}{\mathrm{R}^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (82).$$

Writing $\mathrm{R} = a$, and neglecting a in comparison with c

$$\frac{m}{2\pi c}\left(\log\frac{8c}{\mathrm{R}} - \frac{1}{4}\right)\sin\chi - \dot{z}_0 \sin\chi + \dot{c}\cos\chi + \frac{m}{\pi a}\Sigma(\alpha_n \cos n\chi - \beta_n \sin n\chi)$$

$$= \dot{a} + a\Sigma(\dot{\alpha}_n \sin n\chi + \dot{\beta}_n \cos n\chi) + \frac{m}{\pi a}\Sigma n(\alpha_n \cos n\chi - \beta_n \sin n\chi) \quad . \quad (83).$$

This equation gives

$$\dot{a} = 0, \quad \dot{c} = 0 \qquad \dot{z}_0 \text{ or } \mathrm{V} = \frac{m}{2\pi c}\left(\log\frac{8c}{a} - \frac{1}{4}\right)$$

$$\left.\begin{aligned} \dot{\alpha}_n - \frac{m}{\pi a^2}(n-1)\beta_n = 0 \\ \dot{\beta}_n + \frac{m}{\pi a^2}(n-1)\alpha_n = 0 \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (84).$$

Therefore

$$\ddot{\alpha}_n + \frac{m^2}{\pi^2 a^4}(n-1)^2 \alpha_n = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (85).$$

Therefore the time of an oscillation of the type

$$\alpha_n \sin n\chi + \beta_n \cos n\chi$$

is

$$\frac{2\pi}{\dfrac{m}{\pi a^2}(n-1)} = \frac{2\pi^2 a^2}{m(n-1)}$$

The steady motion and small fluted oscillations of a single vortex ring have been worked out by Mr. HICKS by means of toroidal functions. Only the simplest case, that of a vortex ring in a fluid of equal density, with no added circulations, is considered in this paper; the same methods might be applied to other cases. The steady motion given above agrees with Mr. HICKS' result, in the velocity V, and the value of β_n to the first order.

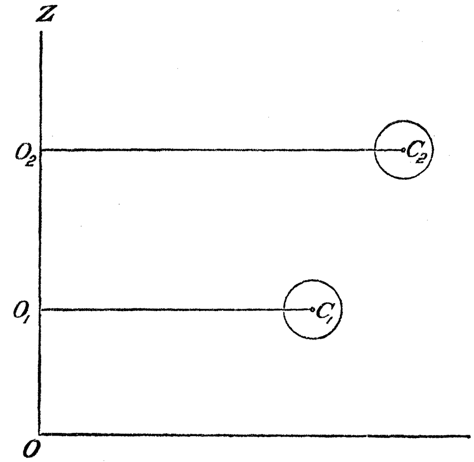
SECTION V.

The motion of any number of fine Vortex rings on the same axis.

§ 38. *The stream-line function.*

The cross-sections of the rings are approximately circles.

In each ring, let the vorticity be constant. Then $\omega/\varpi =$ constant, over each ring.



Let $\omega = \frac{1}{2}M_1\varpi,\ \frac{1}{2}M_2\varpi,\ \frac{1}{2}M_3\varpi$, &c., in the several rings.

The stream-function Ψ satisfies

$$\frac{d^2\Psi}{dz^2} + \frac{d^2\Psi}{d\varpi^2} - \frac{1}{\varpi}\frac{d\Psi}{d\varpi} = 0, \text{ outside the rings ;}$$

$$\frac{d^2\Psi}{dz^2} + \frac{d^2\Psi}{d\varpi^2} - \frac{1}{\varpi}\frac{d\Psi}{d\varpi} + M_1\varpi^2 = 0, \text{ inside the ring } C_1\text{ ;}$$

$$\frac{d^2\Psi}{dz^2} + \frac{d^2\Psi}{d\varpi^2} - \frac{1}{\varpi}\frac{d\Psi}{d\varpi} + M_2\varpi^2 = 0, \text{ inside the ring } C_2, \text{ \&c.;}$$

while Ψ and $d\Psi/dn$ are continuous everywhere.

The value of Ψ at any point ϖ', z', ϕ', outside the rings is given by

$$\Psi = \varpi' \Sigma \frac{M_1}{4\pi} \iiint \frac{\varpi^2 \cos\phi \, dz \, d\varpi \, d\phi}{\sqrt{\{(z'-z)^2 + \varpi'^2 - 2\varpi\varpi'\cos\phi + \varpi'^2\}}} \quad . \quad . \quad . \quad . \quad (86),$$

where the integrals are taken throughout the volume of each ring.

Consider the integral over the ring whose mean radius is c_1, and cross-section a_1, and whose central circle is distant z_1 from the plane of xy.

Let

$$\left.\begin{aligned} \varpi &= c_1 - f \\ z &= z_1 + h \end{aligned}\right\}.$$

Due to this ring alone

$$\Psi = \frac{M_1\varpi'}{4\pi}\iint e^{-f(d/dc_1)+h(d/dz_1)}\,df\,dh\int_0^{2\pi}\frac{c_1^2\cos\phi\,d\phi}{\sqrt{\{(z'-z_1)^2+\varpi'^2-2c_1\varpi'\cos\phi+c_1^2\}}}$$

$$= \tfrac{1}{2}M_1a_1^2\,\varpi'\int_0^{\pi}\frac{c_1^2\cos\phi\,d\phi}{\sqrt{\{(z'-z_1)^2+\varpi'^2-2c_1\varpi'\cos\phi+c_1^2\}}}\quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (87)$$

Take a point on the surface of the first ring. Let its coordinates be

$$c_1 - a_1\cos\chi,\quad z_1 + a_1\sin\chi.$$

The part of Ψ due to the ring itself is

$$\frac{M_1a_1^2c_1^2}{2}\left\{\log\frac{8c_1}{a_1} - 2 + \frac{\log 8c_1/a_1 - \frac{1}{4}}{2}\frac{a_1}{c_1}\cos\chi + \ldots\right\}\quad . \quad . \quad . \quad . \quad (88).$$

The part due to the ring C_2 is

$$\frac{M_2a_2^2}{2}\int_0^{\pi(c_1-a_1\cos\chi)}\frac{c_2^2\cos\phi\,d\phi}{\sqrt{\{(z_1+a_1\sin\chi-z_2)^2+(c_1-a_1\cos\chi)^2-2c_2(c_1-a_1\cos\chi)\cos\phi+c_2^2\}}}.$$

Let the integral

$$\int_0^{\pi}\frac{c_1c_2\cos\phi\,d\phi}{\sqrt{\{(z_1-z_1)^2+c_1^2-2c_1c_2\cos\phi+c_2^2\}}}$$

be called I_{12}.

Then the part of Ψ due to the ring C_2 is

$$\frac{M_2a_2^2c_2}{2}\left\{I_{12} + a_1\sin\chi\frac{dI_{12}}{dz_1} - a_1\cos\chi\frac{dI_{12}}{dc_1}\right\}\quad . \quad . \quad . \quad . \quad . \quad . \quad (89).$$

a^2c_1 is constant: write

$$\tfrac{1}{2}M_1\,a_1^2c_1 = \frac{m_1}{\pi}.$$

Then at a point $c_1 - a_1\cos\chi$, $z_1 + a_1\sin\chi$, on the surface of the ring C_1, we find

$$\Psi = \left.\begin{aligned}&\frac{m_1}{\pi}\left\{c_1\log\frac{8c_1}{a_1} - 2 - \frac{\log 8c_1/a_1 - \frac{1}{4}}{2}\frac{a_1}{c_1}\cos\chi + \ldots\right\}\\ &+\frac{m_2}{\pi}\left\{I_{12} + a_1\sin\chi\frac{dI_{12}}{dz_1} - a_1\cos\chi\frac{dI_{12}}{dc_1} + \ldots\right\}\\ &+\frac{m_3}{\pi}\left\{I_{13} + a_1\sin\chi\frac{dI_{13}}{dz_1} - a_1\cos\chi\frac{dI_{13}}{dc_1} + \ldots\right\}\\ &+\ldots\end{aligned}\right\}\quad . \quad . \quad (90).$$

§ 39. *The equations of Motion.*

Let the ring C_1 be moving forward with velocity $\dot{z}_1$

Let its radius increase with velocity $\dot{c}_1$. The radius of the cross-section will change, but since $a_1^2 c_1 = \text{const.}$,

$$\dot{a}_1 = -\tfrac{1}{2}\frac{a_1}{c_1}\dot{c}_1,$$

and therefore (a_1/c_1 being small), $\dot{a}_1$ is negligible compared with $\dot{c}_1$.

The normal velocity of any point on the ring's surface is

$$\dot{z}_1 \sin\chi - \dot{c}_1 \cos\chi - \dot{a}_1.$$

But the resolved part of the velocity along the normal to the ring is

$$\frac{1}{\varpi}\frac{d\Psi}{d\varpi}\frac{d\varpi}{ds} + \frac{1}{\varpi}\frac{d\Psi}{dz}\frac{dz}{ds}.$$

Therefore

$$\frac{d\Psi}{ds} = \varpi(\dot{z}_1 \sin\chi - \dot{c}_1 \cos\chi - \dot{a}_1) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (91)$$

at the surface of the ring.

Therefore at the surface of the ring C_1,

$$\begin{aligned}\Psi &= \int(\dot{z}_1 \sin\chi - \dot{c}_1 \cos\chi - \dot{a}_1)\,\varpi\, ds \\ &= \int(\dot{z}_1 \sin\chi - \dot{c}_1 \cos\chi - \dot{a}_1)(c_1 - a_1\cos\chi)\,a_1\,d\chi \\ &= -a_1 c_1 \dot{z}_1 \cos\chi - a_1 c_1 \dot{c}_1 \sin\chi + \text{terms of higher orders} \quad . \quad . \quad (92).\end{aligned}$$

Comparing this with the value already found for Ψ at the surface of the ring, we obtain

$$\left.\begin{aligned} c_1\dot{z}_1 &= \frac{m_1}{2\pi}\left(\log\frac{8c_1}{a_1} - \tfrac{1}{4}\right) + \frac{m_2}{\pi}\frac{dI_{12}}{dc_1} + \frac{m_3}{\pi}\frac{dI_{13}}{dc_1} + \ldots \\ \text{and } -c_1\dot{c}_1 &= \frac{m_2}{\pi}\frac{dI_{12}}{dz_1} + \frac{m_3}{\pi}\frac{dI_{13}}{dz_1} + \ldots \end{aligned}\right\} \quad . \quad . \quad . \quad (93).$$

If we write

$$\Sigma\frac{m_1 m_2}{\pi} I_{12} = U,$$

then this equation may be written

$$\left.\begin{aligned} m_1 c_1 \dot{z}_1 &= \frac{m_1^2}{2\pi}\log\left(\frac{8c_1}{a_1} - \tfrac{1}{4}\right) + \frac{dU}{dc_1} \\ \text{and}\quad -m_1 c_1 \dot{c}_1 &= \frac{dU}{dz_1} \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad (94).$$

Similar equations hold for each of the other rings.

§ 40. *Two integrals of the equations.*

Since U is a function of $z_1 - z_2$, $z_1 - z_3$, $z_2 - z_3$, &c.,

$$\frac{dU}{dz_1} + \frac{dU}{dz_2} + \ldots = 0.$$

Thus $\Sigma\, m_1 c_1 \dot{c}_1 = 0$.

Therefore

$$\Sigma\, m_1 c_1^2 = \text{const.} \qquad (95).$$

Again, multiply the equations by $\dot{c}_1$, $\dot{z}_1$, $\dot{c}_2$, $\dot{z}_2$, &c., and add.

Therefore

$$\Sigma\, \frac{M_1^2}{2\pi}\left(\log\frac{8c_1}{a_1} - \tfrac{1}{4}\right)\dot{c}_1 + \Sigma\left(\frac{dU}{dc_1}\dot{c}_1 + \frac{dU}{dz_1}\dot{z}_1\right) = 0. \qquad (96).$$

Remembering that $a_1^2 c_1 = \text{const.}$; this equation gives on integration

$$\Sigma\, \frac{M_1^2}{2\pi c_1}\left(\log\frac{8c_1}{a_1} - \tfrac{7}{4}\right) + U = \text{const.} \qquad (97).$$

§ 41. These integrals are the equations expressing the constancy of linear momentum and of energy in the system.

They are easily obtained directly.

The momentum

$$= \iiint \frac{1}{\varpi}\frac{d\Psi}{d\varpi}\, d\varpi\, dz\, d\phi \qquad (98),$$

the integral being taken all over space.

Since Ψ and $d\Psi/d\varpi$ are continuous everywhere, this

$$= 2\pi \int_{-\infty}^{\infty} (\Psi_\infty - \Psi_0)\, dz \qquad (99),$$

where Ψ_∞ = value of Ψ at a great distance from the axis, and Ψ_0 = value of Ψ at the axis.

Now

$$\Psi_\infty = \Sigma\, \frac{M_1}{\pi}\int_0^\pi \frac{\varpi c_1 \cos\phi\, d\phi}{\sqrt{(z - z_1)^2 + \varpi^2 - 2\,\varpi c_1 \cos\phi + c_1^2}}$$

$$= \Sigma\, \frac{M_1}{2}\, \frac{\varpi^2 c_1^2}{\{(z - z_1)^2 + \varpi^2\}^{\frac{3}{2}}} \qquad (100),$$

and

$$\Psi_0 = 0.$$

Therefore the momentum

$$= 2\pi \Sigma \frac{m_1}{2} \int_{-\infty}^{\infty} \frac{c_1^2 \varpi^2 \, dz}{\{(z - z_1)^2 + \varpi^2\}^{\frac{3}{2}}}$$

$$= 2\pi \Sigma m_1 c_1^2 \int_0^{\pi/2} \frac{\sec^2 \theta \, d\theta}{\sec^3 \theta}, \text{ where } \tan \theta = \frac{z - z_1}{\varpi}$$

$$= 2\pi \Sigma m_1 c_1^2 \quad \ldots \quad (101).$$

§ 42. The kinetic energy is given by

$$\mathrm{T} = \pi \iint \frac{1}{\varpi} \left\{ \left(\frac{d\Psi}{d\varpi}\right)^2 + \left(\frac{d\Psi}{dz}\right)^2 \right\} d\varpi \, dz$$

$$= -2\pi \iint \frac{\Psi}{\varpi} \left(\frac{d^2\Psi}{dz^2} + \frac{d^2\Psi}{d\varpi^2} - \frac{1}{\varpi}\frac{d\Psi}{d\varpi}\right) d\varpi \, dz \quad \ldots \quad (102).$$

Now, outside the ring

$$\frac{d^2\Psi}{dz^2} + \frac{d^2\Psi}{d\varpi^2} - \frac{1}{\varpi}\frac{d\Psi}{d\varpi} = 0,$$

and inside the ring

$$= -\frac{2m_1 c_1}{\pi a_1^2}.$$

At the surface of the ring

$$\Psi = \frac{m_1 c_1}{\pi}\left(\log \frac{8c_1}{a_1} - 2\right) + \Sigma \frac{m_2}{\pi} \mathrm{I}_{12} \quad \ldots \quad (103).$$

Therefore, inside the ring C_1,

$$\Psi = \frac{m_1 c_1}{\pi}\left(\log \frac{8c_1}{a_1} - 2\right) + \Sigma \frac{m_2}{\pi} \mathrm{I}_{12} + \frac{m_1 c_1}{2\pi}\left(1 - \frac{\mathrm{R}^2}{a_1^2}\right). \quad \ldots \quad (104).$$

Integrating throughout this ring, the part of the kinetic energy arising from this is

$$\left\{\Sigma m_1^2 c_1 \left(\log \frac{8c_1}{a_1} - \tfrac{7}{4}\right) + m_1 (m_2 \mathrm{I}_{12} + m_3 \mathrm{I}_{13} + \ldots)\right\} \quad \ldots \quad (105).$$

Adding the integrals arising from the other rings, the kinetic energy

$$= 4\pi \left\{\Sigma m_1^2 c_1 \left(\log \frac{8c_1}{a_1} - \tfrac{7}{4}\right) + 2\Sigma m_1 m_2 \mathrm{I}_{12}\right\}. \quad \ldots \quad (106).$$

The equations of motion may be written in the interesting forms,

$$\left.\begin{aligned} m_1 c_1 \dot{z}_1 &= \frac{1}{8\pi}\frac{\partial \mathrm{T}}{\partial c_1} \\ -\, m_1 c_1 \dot{c}_1 &= \frac{1}{8\pi}\frac{\partial \mathrm{T}}{\partial z_1} \end{aligned}\right\} \quad \ldots \quad (107).$$

The preceding work exemplifies an interesting fact about vortex motion. When the configuration of the vortices are known, their various velocities are obtained by linear equations; just as in the case of gravitating matter, linear equations give the accelerations of the various parts.

§ 43. When there are only two rings, the equations of motion are—

$$\left.\begin{aligned} \pi c_1\dot{z}_1 &= \frac{m_1}{2}\left(\log\frac{8c_1}{a_1} - \tfrac{1}{4}\right) + m_2\frac{dI}{dc_1} \\ \pi c_1\dot{c}_1 &= \qquad\qquad\qquad\qquad - m_2\frac{dI}{dz_1} \\ \pi c_2\dot{z}_2 &= \frac{m_2}{2}\left(\log\frac{8c_2}{a_2} - \tfrac{1}{4}\right) + m_1\frac{dI}{dc_2} \\ \pi c_2\dot{c}_2 &= \qquad\qquad\qquad\qquad - m_1\frac{dI}{dz_2} \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (108),$$

where

$$I = \int_0^\pi \frac{c_1c_2\cos\phi\, d\phi}{\sqrt{\{(z_2 - z_1)^2 + c_1^2 - 2c_1c_2\cos\phi + c_2^2\}}}.$$

The two integrals of these equations, already found, are

$$m_1c_1^2 + m_2c_2^2 = \text{constant} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (109),$$

$$\frac{m_1^2}{2}c_1\left(\log\frac{8c_1}{a_1} - \tfrac{7}{4}\right) + \frac{m_2^2}{2}c_2\left(\log\frac{8c_2}{a_2} - \tfrac{7}{4}\right) + m_1m_2 I = \text{constant} \quad . \quad (110).$$

Suppose the ring (C_1) in front of the ring (C_2).

Then $z_1 - z_2$ is positive, and

$$\frac{dI}{dz} = -\int_0^\pi \frac{c_1c_2(z_1 - z_2)\cos\phi\, d\phi}{\sqrt{\{(z_1 - z_2)^2 + c_1^2 - 2c_1c_2\cos\phi + c_2^2\}}}.$$

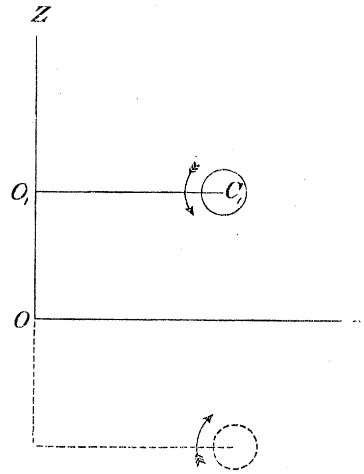
The integral is easily seen to be positive.

Therefore the radius of the front ring increases, and of the back ring diminishes. These changes cause decrease and increase of the velocities, so that it may happen that the second ring will overtake the first.

That the motion is of this character was shown by HELMHOLTZ, 'CRELLE'S Journal,' vol. 55, pp. 54, 55.

§ 44. *Vortex Ring approaching a Fixed Plane.*

This question is at once reduced to the case of two vortex rings on the same axis, by taking the image of the ring in the plane.



Let z be its distance from the plane at any moment.

The equation of energy gives

$$c\left(\log\frac{8c}{a}-\tfrac{7}{4}\right)-\int_0^\pi\frac{c^2\cos\phi\,d\phi}{\sqrt{\{4z^2+4c^2\sin^2(\tfrac{1}{2}\phi)\}}}=c_0\left(\log\frac{8c_0}{a_0}-\tfrac{7}{4}\right),$$

where c_0, a_0, are the values of c and a when the ring is at a great distance from the plane.

Put

$$\phi=\frac{\pi}{2}-2\psi.$$

Then

$$c\left(\log\frac{8c}{a}-\tfrac{7}{4}\right)-\sqrt{z^2+c^2}\,(\mathrm{F}-\mathrm{E})+\frac{c^2}{\sqrt{z^2+c^2}}\,\mathrm{F}=c_0\left(\log\frac{8c_0}{a_0}-\tfrac{7}{4}\right)\quad .\quad (111),$$

where $\dfrac{c}{\sqrt{c^2+z^2}}$ is the modulus of the elliptic functions.

This equation gives the relation between the radius of the ring and its distance from the plane.

Put

$$\frac{c}{\sqrt{c^2+z^2}}=\sin\theta.$$

Then

$$c\left(\log\frac{8c}{a}-\tfrac{7}{4}\right)-2c\operatorname{cosec}\theta\,(\mathrm{F}-\mathrm{E})+c\sin\theta\,.\,\mathrm{F}=c_0\left(\log\frac{8c_0}{a_0}-\tfrac{7}{4}\right). \quad (112).$$

In addition to this $a^2c = a_0^2c_0$.

When the ring is so near the plane that z is much smaller than c, though greater than a,

$$\mathrm{F}=\log\frac{4\sqrt{c^2+z^2}}{z}\log\frac{4c}{z}\text{ . approximately.}$$

$$\mathrm{E}=1.$$

Therefore,

$$c\left(\log\frac{2z}{a}+\tfrac{1}{4}\right)=c_0\left(\log\frac{8c_0}{a_0}-\tfrac{7}{4}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (113).$$

When the ring is near the plane, the equations of motion are

$$\dot{c}=\frac{m}{2\pi z} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (114),$$

$$c\dot{z}=-\frac{m}{2}\left(\log\frac{2z}{a}+\tfrac{3}{4}\right) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (115).$$

From these, or more simply from the equation of energy

$$c=\frac{m}{\pi a_0}e^{\frac{1}{4}}\sqrt{\frac{c}{c_0}}\,.\,e^{-c_0/c\,[\log(8c_0/a_0)-\frac{7}{4}]}.$$

Therefore

$$\frac{me^{\frac{1}{4}}}{\pi a_0}(t_2-t_1)=\int_{c_1}^{c_0}\sqrt{\frac{c}{c_0}}\,.\,e^{c_0/c\,[\log(8c_0/a_0)-\frac{7}{4}]}\,dc.$$

Let

$$c/c_0=x^2,$$

and

$$k=\log\frac{8c_0}{a_0}-\tfrac{7}{4}.$$

Then

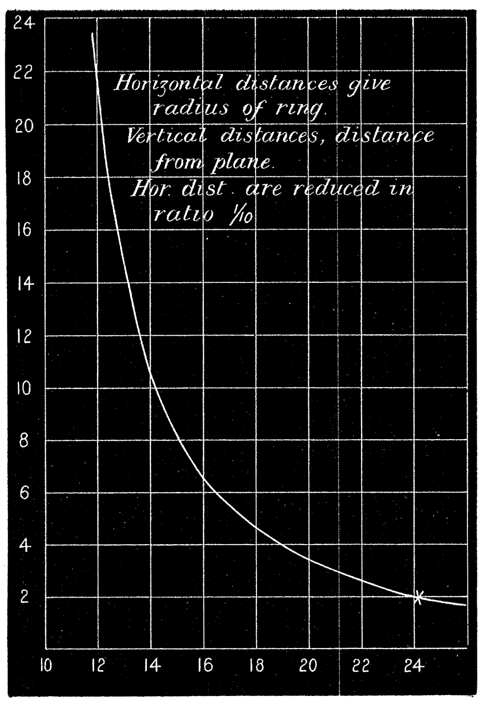
$$\frac{me^{\frac{1}{4}}}{2a_0c_0}(t_2-t_1)=\int_{x_1}^{x_2}\frac{e^{-kx^2}}{x}\,dx \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (116).$$

The following Table shows the change in the radius of a ring in which $c_0/a_0 = 100$. When the ring is near the plane, the approximate formula

$$c\,(\log\frac{2z}{a} + \tfrac{1}{4}) = c_0\,(\log\frac{8c_0}{a_0} - \tfrac{7}{4})$$

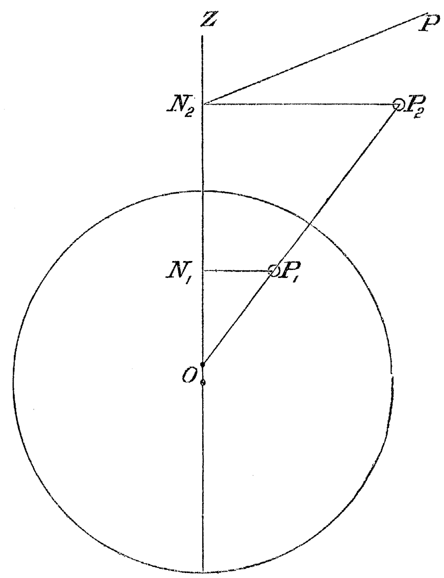
was used, but in other cases the exact formula.

θ.	$\frac{z}{a}$.	$\frac{z}{a_0}$.	$\frac{c}{c_0}$.
45°		104	1·04
60		61·8	1·07
70		40·0	1·10
75		30·5	1·14
80		21·3	1·21
83		16·0	1·30
85		12·1	1·38
86		10·1	1·44
	10	8·12	1·52
	8	6·26	1·63
	6	4·67	1·80
	5	3·60	1·93
	4	2·75	2·12
	3	1·93	2·42



§ 45. *Vortex ring passing over a Fixed Spherical Obstacle.*

When a vortex ring lies on a sphere concentric with a sphere in the fluid, the method of images may be applied, as is shown by Mr. LEWIS, 'Q.J.M.,' vol. 16, p. 338.



Let m_2 be the strength of the external ring

c_2 its radius,

a_2 the radius of its cross-section,

z_2 the distance of its centre along OZ.

Let m_1, c_1, a_1, z_1, be corresponding quantities for the image

Then

$$m_1 = -m_2 \sqrt{\frac{c_2}{c_1}},$$

$$\frac{c_1}{a} = \frac{c_2}{a_2}.$$

At any external point P (coor. z, ϖ, ϕ),

$$\Psi = m_2 \int_0^\pi \frac{c_2 \varpi \cos\phi \, d\phi}{\sqrt{\{(z-z_2)^2 + \varpi^2 - 2\varpi c_2 \cos\phi + c_2^2\}}}$$

$$- m_2 \sqrt{\frac{c_2}{c_1}} \int_0^\pi \frac{c_1 \varpi \cos\phi \, d\phi}{\sqrt{\{(z-z_1)^2 + \varpi^2 - 2\varpi c_1 \cos\phi + c_1^2\}}}.$$

The equation of energy becomes

$$\frac{m_2^2c_2}{2}\left(\log\frac{8c_2}{a_2}-\tfrac{7}{4}\right)+\frac{m_2^2c_2}{c_1}\frac{c_1}{2}\log\left(\frac{8c_1}{a_1}-\tfrac{7}{4}\right)$$

$$-m_2^2\sqrt{\frac{c_2}{c_1}}\int_0^\pi\frac{c_1c_2\cos\phi\,d\phi}{\sqrt{\{(z_2-z_1)^2+c_2^2-2c_1c_2\cos\phi+c_1^2\}}}$$

$$=m_2^2c_0\left(\log\frac{8c_0}{a_0}-\tfrac{7}{4}\right)\quad\ldots\ldots\quad(117),$$

where a_0 and c_0 are the values of a and c, when the ring and sphere are a long way apart.

Let the radius of the sphere be k.

Changing to polar coordinates, let

$$c_2=r\sin\theta.\qquad c_1=\frac{k^2}{r}\sin\theta.$$

$$z_2=r\cos\theta.\qquad c_2=\frac{k^2}{r}\cos\theta.$$

Then

$$(z_2-z_1)^2+c_2^2-2c_1c_2\cos\phi+c_1^2=r^2+\frac{k^4}{r^2}-2k^2\cos^2\theta-2k^2\sin^2\theta\cos^2\phi$$

$$=\left(r-\frac{k^2}{r}\right)^2+4k^2\sin\theta\sin^2\frac{\phi}{2}.$$

Therefore

$$r\sin\theta\left(\log\frac{8r\sin\theta}{a}-\tfrac{7}{4}\right)-rk\sin^2\theta\int_0^\pi\frac{\cos\phi\,d\phi}{\sqrt{\{(r-k^2/r)^2+4k^2\sin^2\theta\sin^2(\frac{1}{2}\phi)\}}}$$

$$=r_0\sin\theta_0\left(\log\frac{8r_0\sin\theta_0}{a_0}-\tfrac{7}{4}\right).$$

Also

$$a^2r\sin\theta=a_0^2r_0\sin\theta_0.$$

Substituting a from the second equation, we have the path of the ring from infinity up to the sphere, *i.e.*, the path of any point in the ring.

Let

$$r_0\sin\theta_0=c_0\quad\text{and}\quad c_0/a_0=n.$$

Then the above equation becomes

$$r\sin\theta\left\{\log 8n-\tfrac{7}{4}+\tfrac{3}{2}\log\frac{r\sin\theta}{c_0}\right\}-kr\sin^2\theta\int_0^\pi\frac{\cos\phi\,d\phi}{\sqrt{\{(r-k^2/r)^2+4k^2\sin^2\theta\sin^2(\frac{1}{2}\phi)\}}}$$

$$=c_0(\log 8n-\tfrac{7}{4})\quad\ldots\ldots\quad(118).$$

When the centre of the ring coincides with the centre of the sphere, let $r\sin\theta=\varpi$, $\sin\theta=1$; and an easy transformation of the integral gives

$$\varpi\left\{\log 8n - \tfrac{7}{4} + \tfrac{3}{2}\log\frac{\varpi}{c_0}\right\} - 2k^3\int_0^{\pi/2}\frac{\sin^2\psi\,d\psi}{\sqrt{\varpi^4 - k^4\sin^2\psi}} = c_0\left(\log 8n - \tfrac{7}{4}\right).$$

Therefore

$$\frac{\varpi}{c_0}\left\{\log 8n - \tfrac{7}{4} + \tfrac{3}{2}\log\frac{\varpi}{c_0}\right\} - 2\frac{\varpi}{k}\frac{\varpi}{c_0}(\mathrm{F} - \mathrm{E}) - \left(\log 8n - \tfrac{7}{4}\right) = 0,$$

where the modulus of the elliptic function is k^2/ϖ^2.

Therefore

$$\frac{\mathrm{F} - \mathrm{E}}{\sqrt{\sin\alpha}} = \tfrac{1}{2}\left(\log 8n - \tfrac{7}{4} + \tfrac{3}{2}\log\frac{\varpi}{c_0}\right) - \tfrac{1}{2}\frac{c_0}{\varpi}\left(\log 8n - \tfrac{7}{4}\right) \quad . \quad . \quad . \quad (119),$$

where $\sin\alpha = k^2/\varpi^2$.

Let c_0/a_0 or $n = 100$.

Then

$$\frac{\mathrm{F} - \mathrm{E}}{\sqrt{\sin\alpha}} = 2{\cdot}467\left(1 - \frac{c_0}{\varpi}\right) + \tfrac{3}{4}\log_{10}\frac{\varpi}{c_0}.$$

It is easy to construct the following table :—

α.	ϖ.	c_0.	k.
20°	1	·98	·58
30	1	·90	·71
45	1	·82	·84
60	1	·70	·93

From which, taking c_0 the value of the radius at infinity as the unit, we see that when k (radius of sphere) $=$ ·6, ·8, 1, 1·3; ϖ, the radius of the ring when passing the middle of the sphere, $=$ 1·02, 1·1, 1·3, 1·4.

[* § 46. *Two coaxal rings of equal strength and volume moving in the same direction.* Let the radii of the rings be c_1 and c_2, and the radii of their cross-sections a_1 and a_2. Then

$$a_1^2 c_1 = a_2^2 c_2 = \text{constant} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (120).$$

The equation of momentum is

$$c_1^2 + c_2^2 = 2\kappa^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (121),$$

and the equation of energy is

* This and the next paragraph have been altered since the paper was read, in consequence of a very valuable suggestion of one of the Referees.—July, 1893.

$$\frac{c_1}{2}\left(\log\frac{8c_1}{a_1}-\tfrac{7}{4}\right)+\frac{c_2}{2}\left(\log\frac{8c_2}{a_2}-\tfrac{7}{4}\right)+\int_0^\pi\frac{c_1c_2\cos\phi\,d\phi}{\sqrt{\{(z_1-z_2)^2+c_1^2-2c_1c_2\cos\phi+c_2^2\}}}=\text{constant (122)},$$

and if m be the strength of each ring,

$$\dot{z}_1-\dot{z}_2=\frac{m}{\pi}\left\{\frac{1}{2c_1}\left(\log\frac{8c_1}{a_1}-\tfrac{1}{4}\right)-\frac{1}{2c_2}\left(\log\frac{8c_2}{a_2}-\tfrac{1}{4}\right)+\frac{1}{c_1}\frac{dI}{dc_1}-\frac{1}{c_2}\frac{dI}{dc_2}\right\}.$$

Let us write $z_1-z_2=z$.

Then

$$\frac{1}{c_1}\frac{dI}{dc_1}-\frac{1}{c_2}\frac{dI}{dc_2}$$

$$=\int_0^\pi\frac{(c_2^2-c_1c_2\cos\phi+z^2)\,c_2\cos\phi\,d\phi}{c_1(c_1^2-2c_1c_2\cos\phi+c_2^2+z^2)^{\frac{3}{2}}}-\int_0^\pi\frac{(c_1^2-c_1c_2\cos\phi+z^2)\,c_1\cos\phi\,d\phi}{c_1(c_1^2-2c_1c_2\cos\phi+c_2^2+z^2)^{\frac{3}{2}}}$$

$$=\frac{c_2^2-c_1^2}{c_1c_2}\int_0^\pi\frac{(c_1^2-c_1c_2\cos\phi+c^2+z^2)\cos\phi\,d\phi}{(c_1^2-2c_1c_2\cos\phi+c_2^2+z^2)^{\frac{3}{2}}}$$

$$=\frac{c_2^2-c_1^2}{c_1c_2}\int_0^\pi\frac{\cos\phi\,d\phi}{(c_1^2-2c_1c^2\cos\phi+c_2^2+z^2)^{\frac{1}{2}}}+(c_2^2-c_1^2)\int_0^\pi\frac{(1-\sin^2\phi)\,d\phi}{(c_1^2-2c_1c_2\cos\phi+c_2^2+z^2)^{\frac{3}{2}}}$$

$$=\int_0^\pi\frac{(c_2^2-c_1^2)\,d\phi}{(c_1^2-2c_1c_2\cos\phi+c_2^2+z^2)^{\frac{3}{2}}}.$$

Therefore

$$\dot{z}=\frac{m}{\pi}\left\{\frac{1}{2c_1}\left(\log\frac{8c_1}{a_1}-\tfrac{1}{4}\right)-\frac{1}{2c_2}\left(\log\frac{8c_2}{a_2}-\tfrac{1}{4}\right)+(c_2^2-c_1^2)\int_0^\pi\frac{d\phi}{(c_1^2-2c_1c_2\cos\phi+c_2^2+z^2)^{\frac{3}{2}}}\right\}\quad(123).$$

Now the rings are at their greatest distance apart when $\dot{z}_1=\dot{z}_2$ or when $\dot{z}$ vanishes. Now, equation (123) shows that this takes place when $c_1=c_2$.

When $c_1=c_2$ let the value of each be κ, and let a_1 and a_2 each $=\alpha$. Also let κ_1, $\alpha_1:\kappa_2$, α_2, be the values of c_1, $a_1:c_2$, a_2, when $z_1=z_2$, *i.e.*, when the rings are in the same plane.

Then these quantities are connected by the equations

$$\alpha_1^2\kappa_1=\alpha_2^2\kappa^2=\alpha^2\kappa,$$

$$\kappa_1^2+\kappa_2^2=2\kappa^2,$$

and

$$\frac{\kappa_1}{2}\left(\log\frac{8\kappa_1}{a_1}-\tfrac{7}{4}\right)+\frac{\kappa_2}{2}\left(\log\frac{8\kappa_2}{\alpha_2}-\tfrac{7}{4}\right)+\int_0^\pi\frac{\kappa_1\kappa_2\cos\phi\,d\phi}{\sqrt{\{\kappa_1^2-2\kappa_1\kappa_2\cos\phi+\kappa_2^2\}}}$$

$$=\kappa\left(\log\frac{8\kappa}{\alpha}-\tfrac{7}{4}\right)+\int_0^\pi\frac{\kappa^2\cos\phi\,d\phi}{\sqrt{\{2\kappa^2+z^2-2\kappa^2\cos\phi\}}}.$$

Thus, when we are given κ_1 and κ_2, we have two equations to determine κ and z; or given z and κ, the greatest distance apart of the two rings and their radius at that time, we have equations to find κ_1 and κ_2.

One of the quantities, α_1, α_2, α, is arbitrary, being the cross-section of the ring, when it has a definite radius.

Let us take

$$\alpha = \frac{\kappa}{n}.$$

Also let

$$\kappa_1 = \kappa\sqrt{2}\sin\theta_0 \qquad \text{and} \qquad \kappa_2 = \kappa\sqrt{2}\cos\theta_0.$$

Then the equation connecting z and θ_0 is

$$\frac{\sin\theta_0}{\sqrt{2}}[\log 8n - \tfrac{7}{4} + \tfrac{3}{2}\log(\sqrt{2}\sin\theta_0)] + \frac{\cos\theta_0}{\sqrt{2}}[\log 8n - \tfrac{7}{4} + \tfrac{3}{2}\log(\sqrt{2}\cos\theta_0)]$$
$$+ \frac{1}{\sqrt{2}}\int_0^\pi \frac{\sin 2\theta_0 \cos\phi\, d\phi}{\sqrt{(1 - \sin 2\theta_0\cos\phi)}}$$
$$= \log 8n - \tfrac{7}{4} + \int_0^\pi \frac{\cos\phi\, d\phi}{\sqrt{\left(2 + \frac{z^2}{\kappa^2} - 2\cos\phi\right)}} \quad . \quad . \quad . \quad . \quad . \quad (124).$$

This equation can only be satisfied when θ_0 is between certain limits β and $(\pi/2) - \beta$.

The limiting values of θ_0 give $z = \infty$.

When θ_0 is between β and $(\pi/2) - \beta$, the equation gives a real value of z; this value becomes smaller as θ_0 approaches $\pi/4$.

We have, therefore, the following theorem.

If $\kappa\sqrt{2}\sin\theta_0$ and $\kappa\sqrt{2}\cos\theta_0$ be the radii of two coaxal vortex rings of equal strength and volume when the rings are in the same plane (θ_0 being $< \pi/4$, so that $\kappa\sqrt{2}\sin\theta_0$ is the radius of the inner ring): then these two rings will continue to thread one another in turns, or will separate to an infinite distance according as θ_0 is $>$ or $<$ β, where β is determined by the equation

$$\frac{\sin\beta}{\sqrt{2}}[\log 8n - \tfrac{7}{4} + \tfrac{3}{2}\log(\sqrt{2}\sin\beta)] + \frac{\cos\beta}{\sqrt{2}}[\log 8n - \tfrac{7}{4} + \tfrac{3}{2}\log(\sqrt{2}\cos\beta)]$$
$$+ \frac{1}{\sqrt{2}}\int_0^\pi \frac{\sin 2\beta\cos\phi\, d\phi}{\sqrt{(1 - \sin 2\beta\cos\phi)}} = \log 8n - \tfrac{7}{4} \quad . \quad . \quad . \quad . \quad . \quad . \quad (125).$$

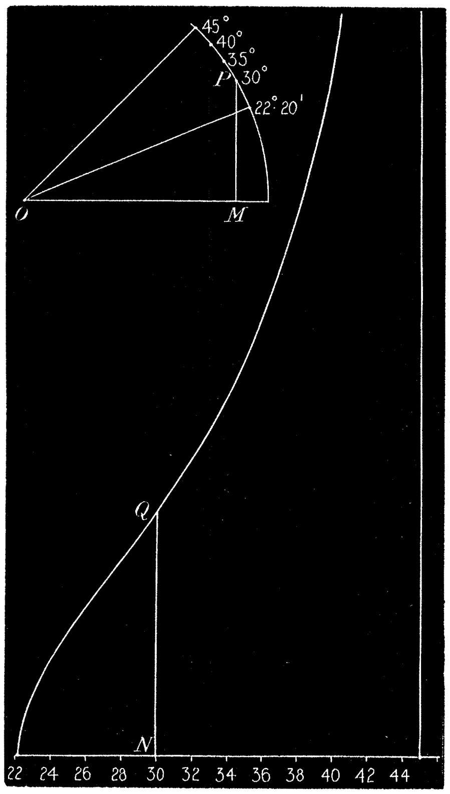
The following Table gives the values of β, and thence of κ_1 and κ_2, for different values of n.

n.	β.	κ_1/κ.	κ_2/κ.
8	18°	·437	1·345
11	19°	·460	1·337
20	20°	·484	1·329
100	22° 20′	·537	1·308
3300	26°	·620	1·271

Roughly speaking, when n is between 8 and 100, the ratio of $\kappa_1 : \kappa_2$ for this limiting case is between $\frac{1}{3}$ and $\frac{2}{5}$.

The following diagram calculated from the equations (121) and (122) gives, for $n=100$, simultaneous values of the radii of the rings and their distance apart in the limiting case in which the inner ring just goes to an infinite distance in front of the outer one.

The radius of the front ring is given by an abscissa of the circle in the diagram, the radius of the back ring by the corresponding ordinate, and the distance apart by the abscissa of the curve. Thus NQ is the distance apart when the ratio of the radii is tan 30°, or when the rings have the radii OM and MP.



§ 47. When θ_0 is greater than the limiting value β, the greatest distance apart to which the vortices can go is given by the equation (124). When θ_0 is less than the

limiting value β, the inner ring goes to an infinite distance from the outer one, increasing in radius towards the limiting value $\kappa\sqrt{2}\sin\theta_\infty$, the radius of the outer ring diminishing towards the limiting value $\kappa\sqrt{2}\sin\theta_\infty$.

The equation determining θ_∞ is

$$\sin\theta_0\{\log 8n - \tfrac{7}{4} + \tfrac{3}{2}\log(\sqrt{2}\sin\theta_0)\} + \cos\theta_0\{\log 8n - \tfrac{7}{4} + \tfrac{3}{2}\log(\sqrt{2}\cos\theta_0)\} + \int_0^\pi \frac{\sin 2\theta_0 \cos\phi\, d\phi}{\sqrt{(1-\sin 2\theta_0\cos\phi)}}$$
$$= \sin\theta_\infty\{\log 8n - \tfrac{7}{4} + \tfrac{3}{2}\log(\sqrt{2}\sin\theta_\infty)\} + \cos\theta_\infty\{\log 8n - \tfrac{7}{4} + \tfrac{3}{2}\log(\sqrt{2}\cos\theta_\infty)\} \quad (126).$$

The table in § 46 gives 22° 20′ as the value for β for rings in which $n = 100$.

The tables below give the greatest distance the rings will separate for values of $\theta_0 > 22° 20'$; and the radii at infinity of the rings for values of $\theta_0 < 22° 20'$. κ_1 and κ_2 are the radii of the rings when they are in the same plane; κ_1', κ_2' are their radii when at an infinite distance apart, if they separate; and z is their greatest distance apart if they remain together.

θ_0.	κ_1.	κ_2.	θ_∞.	κ_1'.	κ_2'.
5°	·123	1·408	5°50′	·144	1·407
10	·246	1·392	12·49	·309	1·379
15	·366	1·366	22·20	·537	1·308
20	·484	1·329	38·30	·880	1·056

θ_0.	κ_1.	κ_2	z.
25°	·598	1·282	1·044
30	·707	1·225	·500
35	·811	1·159	·245
40	·909	1·083	·084